

GLOBAL STRUCTURE OF ADMISSIBLE BV SOLUTIONS TO PIECEWISE GENUINELY NONLINEAR, STRICTLY HYPERBOLIC CONSERVATION LAWS IN ONE SPACE DIMENSION

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ABSTRACT. The paper gives an accurate description of the qualitative structure of an admissible BV solution to a strictly hyperbolic, piecewise genuinely nonlinear system of conservation laws. We prove that there are a countable set Θ which contains all interaction points and a family of countably many Lipschitz curves \mathcal{T} such that outside $\mathcal{T} \cup \Theta$ u is continuous, and along the curves in \mathcal{T} , u has left and right limit except for points in Θ . This extends the corresponding structural result in [8, 10] for admissible solutions.

The proof is based on approximate wave-front tracking solutions and a proper selection of discontinuity curves in the approximate solutions, which converge to curves covering the discontinuities in the exact solution u .

1. INTRODUCTION

This paper is concerned with the qualitative structure of admissible solutions to the strictly hyperbolic $N \times N$ system of conservation laws in one space dimension

$$(1.1) \quad \begin{cases} u_t + f(u)_x = 0 & u : \mathbb{R}^+ \times \mathbb{R} \rightarrow \Omega \subset \mathbb{R}^N, \ f \in C^2(\Omega, \mathbb{R}), \\ u|_{t=0} = u_0 & u_0 \in \text{BV}(\mathbb{R}; \Omega). \end{cases}$$

We assume the strict hyperbolicity in Ω : the eigenvalues $\{\lambda_i(u)\}_{i=1}^N$ of the Jacobin matrix $A(u) = Df(u)$ satisfy

$$\lambda_1(u) < \dots < \lambda_N(u), \quad u \in \Omega.$$

Furthermore, as we only consider the solutions with small total variation, it is not restrictive to assume that Ω is bounded and there exist constants $\{\check{\lambda}_j\}_{j=0}^N$, such that

$$(1.2) \quad \check{\lambda}_{k-1} < \lambda_k(u) < \check{\lambda}_k, \quad \forall u \in \Omega, \ k = 1, \dots, N.$$

Let $\{r_i(u)\}_{i=1}^N$ and $\{l_j(u)\}_{j=1}^N$ be a basis of right and left eigenvectors, depending smoothly on u , such that

$$l_j(u) \cdot r_i(u) = \delta_{ij} \text{ and } |r_i(u)| \equiv 1, \quad i, j = 1, \dots, N.$$

Let $R_i[u_0](\omega)$ be the value at time ω of the solution to the Cauchy problem

$$\frac{du}{d\omega} = r_i(u(\omega)), \quad u(0) = u_0.$$

We call the curve $R_i[u_0]$ the *i-rarefaction curve* through u_0 .

We say that the system (1.1) is *piecewise genuinely nonlinear* if the set where $\nabla \lambda_i \cdot r_i = 0$ is covered by \bar{k}_i transversal manifolds: more precisely,

$$Z_i := \{u \in \Omega : |\nabla \lambda_i \cdot r_i(u)| = 0\} = \bigcup_{j=1}^{\bar{k}_i} Z_i^j,$$

where Z_i^j is a $N - 1$ -dimensional manifolds such that

(1) each Z_i^j is transversal to the vector field $r_i(u)$, i.e.

$$(1.3) \quad (\nabla(\nabla \lambda_i \cdot r_i) \cdot r_i)(u) \neq 0 \quad \text{for } u \in Z_i^j;$$

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(2) each rarefaction curve $R_i[u_0]$ crosses all the Z_i^j , and moreover defining the points $\omega^j[u_0]$ by

$$R_i[u_0](\omega^j[u_0]) \in Z_i^j,$$

then $j \mapsto \omega^j[u_0]$ is strictly increasing.

This implies that along R_i , λ_i has a finite number of critical points.

Denote with Δ_i^j the set of points u between Z_i^j and Z_{i+1}^j :

$$\Delta_i^j := \{u \in \Omega : \omega^j[u] < 0 < \omega^{j+1}[u]\}.$$

Without any loss of generality, we assume that

$$(1.4a) \quad \nabla \lambda_i \cdot r_i(u) < 0 \quad \text{if } j \text{ is even, } u \in \Delta_i^j,$$

$$(1.4b) \quad \nabla \lambda_i \cdot r_i(u) > 0 \quad \text{if } j \text{ is odd, } u \in \Delta_i^j.$$

From now on, we assume that *every characteristic field of (1.1) is piecewise genuinely nonlinear*.

It is well known that, because of the nonlinear dependence of the characteristic speeds $\lambda_i(u)$ on the state variable u , the solution to (1.1) develops discontinuities within finite time, even with smooth initial data. Therefore, in order to construct solutions globally defined in time, one considers weak solutions interpreting the equation (1.1) in a distributional sense. We recall that $u \in C(\mathbb{R}^+; L_{loc}^1(\mathbb{R}; \mathbb{R}^N))$ is a weak solution to the Cauchy problem (1.1) if the initial condition is satisfied and, for any smooth function $\phi \in C_c^1([0, T] \times \mathbb{R})$ there holds

$$\int_0^T \int_{\mathbb{R}} \phi_t(t, x) u(t, x) + \phi_x(t, x) f(u(t, x)) dx dt = 0$$

As a consequence of the weak formulation, it follows that a function with a single jump discontinuity

$$u(t, x) = \begin{cases} u^L & \text{if } x < \sigma t, \\ u^R & \text{if } x > \sigma t, \end{cases}$$

is a solution to (1.1), if and only if the left and right states $u^L, u^R \in \mathbb{R}^N$, and the speed $\hat{\sigma}$ satisfy the Rankine-Hugoniot condition

$$(1.5) \quad f(u^R) - f(u^L) = \hat{\sigma}(u^R - u^L).$$

By the strict hyperbolicity, it is known that for any $u^- \in \Omega$ there exists $s_0 > 0$ and N smooth curves $S_i[u^-] : [-s_0, s_0] \rightarrow \Omega$, associated with functions $\hat{\sigma}_i : [-s_0, s_0] \rightarrow \mathbb{R}$ such that

$$(1.6) \quad \hat{\sigma}_i(s)[S_i[u^-](s) - u^-] = f(\hat{\sigma}_i(S_i[u^-](s))) - f(u^-)$$

and satisfying

$$S_i[u^-](0) = u^-, \quad \hat{\sigma}_i(0) = \lambda_i(u^-), \quad \frac{d}{ds} S_i[u^-](0) = r_i(u^-).$$

The curve $S_i[u^-]$ the i -th *Hugoniot curve* issuing from u^- and we also say that $[u^-, u^+]$ is an i -*discontinuity* with speed $\hat{\sigma}_i(u^-, u^+) := \hat{\sigma}(s)$ if $u^+ = S_i[u^-](s)$.

Since weak solutions to (1.1) may not be unique, an entropy criterion for admissibility is usually added to rule out nonphysical discontinuities. In [10], T.P. Liu proposed the following admissibility criterion valid for weak solutions to general systems of conservation laws

We say the i -discontinuity $[u^-, u^+]$, $u^+ = S_i[u^-](s)$, is *Liu admissible* if it satisfies *Liu admissible condition*: for $s_0 > 0$

$$\hat{\sigma}_i(u^+, u^-) \leq \hat{\sigma}_i(u, u^-)$$

where $u = S_i[u^-](\tau)$ for each $\tau \in]0, s_0[$, and for $s_0 < 0$

$$\hat{\sigma}_i(u^+, u^-) \geq \hat{\sigma}_i(u, u^-),$$

where $u = S_i[u^-](\tau)$ for each $\tau \in]s_0, 0[$.

Let $[u^-, u^+]$, $u^+ = S_i[u^-](s)$, be a Liu admissible i -discontinuity. Following the notation of [10], we call the jump $[u^-, u^+]$ *simple* if $\forall \tau \in]0, s[$ when $s > 0$ ($\forall \tau \in]s, 0[$ when $s < 0$),

$$\hat{\sigma}_i(S_i[u^-](\tau), u^-) < \hat{\sigma}_i(u, u^-) \quad (\hat{\sigma}_i(S_i[u^-](\tau), u^-) > \hat{\sigma}_i(u, u^-)).$$

If $[u^-, u^+]$ is not simple, then we call it a *composition* of the waves $[u^-, u_1], [u_1, u_2], \dots, [u_l, u^+]$, if

$$(1.7) \quad u_k = S_i[u^-](s_k) \quad \text{and} \quad \hat{\sigma}_i(u^k, u^{k-1}) = \hat{\sigma}_i(u^+, u^-),$$

where

$$0 = s_0 < s_1 < s_2 < \cdots < s_l < s \quad (\text{or } s < s_l < \cdots < s_1 < s_0 = 0),$$

and there are no other points τ such that (1.7) holds.

In [10], under assumption of piecewise genuinely nonlinearity, it is proved by using Glimm scheme that if the initial data has small total variation, there exists a weak BV solution of (1.1) satisfying Liu admissible condition. Therefore, it enjoys the usual regularity properties of BV function: u either is approximately continuous or has an approximate jump at each point $(x, t) \in \mathbb{R}^+ \times \mathbb{R} \setminus \mathcal{N}$, where \mathcal{N} is a subset whose one-dimensional Hausdorff measure is zero.

In [10], the author shows much stronger regularity that u holds. The set \mathcal{N} contains at most countably many points. Moreover, u is continuous (not just approximate continuous) outside \mathcal{N} and countably many Lipschitz continuous curves.

In [8], the authors adopt wave-front tracking approximation to prove the similar result for (1.1) with the assumption that each characteristic field is genuinely nonlinear. Moreover, the authors were able to prove that outside the countable set \mathcal{N} there exist right and left limits u^- , u^+ along the jump curves in the uniform norm, and these limits are stable w.r.t. wavefront approximate solutions: more precisely, for each jump point (not interaction point) of the solution, there exists a jump curve for the approximate solution converging to it and such that its left and right limit converge to u^- , and u^+ uniformly. In [6] (Theorem 10.4), the author generalize his result in [8] to the case when some characteristic field may be linearly degenerate.

To prove this new regularity estimates one has to overcome additional difficulties, and this is the reason why they have so far been restricted to genuinely nonlinear or linearly degenerate systems: in fact the proof in [6] is based on the wave structure of the solution to genuinely nonlinear or linearly degenerate systems, where only one shock curve passes through the discontinuous point (which is not an interaction point) of the admissible solution.

In this paper, we extend the techniques of [6] to prove an analogous result about global structure of admissible solution for piecewise genuinely nonlinear system of (1.1) by means of wave-front tracking approximation. This not only completes the corresponding result in [10], but also makes it possible to prove SBV regularity for the solution of piecewise genuinely nonlinear strictly hyperbolic system. In fact, one of the key argument for SBV regularity in the proofs contained in [4] and [5], is that outside the interaction points the left and right values of jumps are approximated uniformly by wavefront approximate solutions.

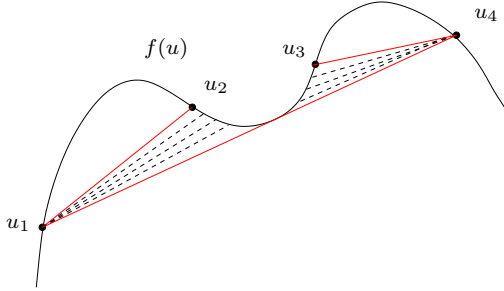


FIGURE 1.

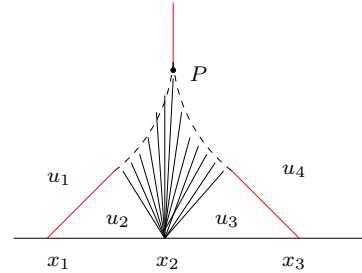


FIGURE 2.

As we said, the assumption of genuinely nonlinearity done in [8] implies that there is only one shock curve passing through the discontinuous point (which is not an interaction point) of the admissible solution. For piecewise genuinely nonlinear case, however, due to the presence of the composite discontinuity, there may be several discontinuity curves passing through the same discontinuity point which is not an interaction point. For example, consider a scalar equation where f has two inflection points (it is thus clearly piecewise genuinely nonlinear), Figure 1, and let u_0 be the initial data be

$$u_0 = \begin{cases} u_1 & \text{if } x < x_1, \\ u_2 & \text{if } x_1 < x < x_2, \\ u_3 & \text{if } x_2 < x < x_3, \\ u_4 & \text{if } x > x_3. \end{cases}$$

The same Figure 2 shows two shocks connecting value u_1, u_2 and u_3, u_4 respectively interact with a center rarefaction wave and eventually have the same speed at P (which is thus *not* an interaction point) and combine together becoming a single shock. Clearly such wave pattern can not happen if f is convex or concave.

In this paper we prove the following theorem:

Theorem 1.1. *Let u be a Liu admissible solution of the Cauchy problem (1.1). Then there exist a countable set Θ of interaction points and a countable family \mathcal{T} of Lipschitz continuous curves such that u is continuous outside Θ and $\text{Graph}(\mathcal{T})$.*

Moreover, suppose $u(t_0, x)$ is discontinuous at $x = x_0$, and $(t_0, x_0) \notin \Theta$. Write $u^L = u(t_0, x_0 -)$, $u^R = u(t_0, x_0 +)$ and suppose that $u^R = S_i[u^L](s)$ with $s > 0$ ($s < 0$).

- If $[u^L, u^R]$ is simple, there exists a Lipschitz curve $y \in \mathcal{T}$, s.t. $y(t_0) = x_0$

$$u^L = \lim_{\substack{x < y(t) \\ (x, t) \rightarrow (t_0, x_0)}} u(x, t), \quad u^R = \lim_{\substack{x > y(t) \\ (x, t) \rightarrow (t_0, x_0)}} u(x, t)$$

and the curve y propagates with shock speed $\hat{\sigma}(u^L, u^R)$ at (t_0, x_0) , that is

$$\dot{y}(t_0)(u^R - u^L) = f(u^R) - f(u^L).$$

and

$$\dot{y}(t_0) \leq \hat{\sigma}_i(S_i[u^L](\tau), u^L), \quad \forall \tau \in [0, s] \quad (\dot{y}(t_0) \geq \hat{\sigma}_i(S_i[u^L](\tau), u^L), \quad \forall \tau \in [s, 0]).$$

- If $[u^L, u^R]$ is a composition of $[u^L, u_1]$, $[u_1, u_2], \dots, [u_l, u^R]$, then there exists p Lipschitz continuous curves $y_1, \dots, y_p \in \mathcal{T}$, $p \leq l + 1$ satisfying
 - $y_1(t_0) = \dots = y_p(t_0) = x_0$,
 - $y'_1(t_0) = \dots = y'_p(t_0)$,
 - $y_1(t) \leq \dots \leq y_p(t)$, for all t in a neighborhood of t_0 ,
 s.t.

$$u^L = \lim_{\substack{x < y_1(t) \\ (x, t) \rightarrow (t_0, x_0)}} u(x, t), \quad u^R = \lim_{\substack{x > y_p(t) \\ (x, t) \rightarrow (t_0, x_0)}} u(x, t),$$

and if in a small neighborhood of (t_0, x_0) , y_j and y_{j+1} are not identical, one has

$$(1.8) \quad u_j = \lim_{\substack{y_j(t) < x < y_{j+1}(t) \\ (x, t) \rightarrow (t_0, x_0)}} u(x, t).$$

Also, these curves propagate with speed $\hat{\sigma}(u^L, u^R)$ at (t_0, x_0) , that is

$$\dot{y}_n(t_0)(u^R - u^L) = f(u^R) - f(u^L), \quad n \in \{1, \dots, p\}.$$

and the stability conditions hold:

$$\dot{y}_n(t_0) \leq \hat{\sigma}_i(S_i[u^L](\tau), u^L), \quad \forall \tau \in [0, s], \quad (\dot{y}_n(t_0) \geq \hat{\sigma}_i(S_i[u^L](\tau), u^L), \quad \forall \tau \in [s, 0]).$$

As in [6], the above result is based on this strong convergence result for approximate wave-front solutions.

Theorem 1.2. *Consider a sequence of wave-front tracking approximate solutions u_ν (see Section 3 for definitions) converging to u in L^1_{loc} . Suppose $P = (\tau, \xi)$ is a discontinuous point of u and write $u^L = u(\tau, \xi -)$, $u^R = u(\tau, \xi +)$. Assume there are only l Lipschitz continuous curves $\mathcal{T} \ni y_n : [t_n^-, t_n^+] \mapsto \mathbb{R}$, $n = 1, \dots, l$ passing through the point P and*

$$y_1(t) \leq \dots \leq y_l(t) \quad \text{in a small neighborhood of } \tau.$$

Then up to a subsequence, there exists $y_{n,\nu} : [t_{n,\nu}^-, t_{n,\nu}^+] \mapsto \mathbb{R}$, $n = 1, \dots, l$ which are discontinuity curves of u_ν with uniformly large strengths, where $t_{n,\nu}^- \rightarrow t_n^-, t_{n,\nu}^+ \rightarrow t_n^+$ and

$$y_{n,\nu}(t) \rightarrow y_n(t) \quad \text{for every } t \in [t_n^-, t_n^+].$$

Moreover, one has

$$\lim_{r \rightarrow 0^+} \limsup_{\nu \rightarrow \infty} \left(\sup_{\substack{x < y_{1,\nu}(t) \\ (x, t) \in B(P, r)}} |u_\nu(x, t) - u^L| \right) = 0,$$

$$\lim_{r \rightarrow 0+} \limsup_{\nu \rightarrow \infty} \left(\sup_{\substack{x > y_{l,\nu}(t) \\ (x,t) \in B(P,r)}} |u_\nu(x,t) - u^R| \right) = 0.$$

A brief outline of this paper follows.

In Section 2, we recall the definition of construction of Riemann solvers introduced in [3].

In Section 3, we briefly describe the wave-front tracking approximate scheme which originally designed for general strictly hyperbolic system (see [1]). In particular, we introduce the definition of interaction and cancelation measures.

Section 4 contains the main idea of the paper: the definition of subdiscontinuity curves and (ϵ, k) -approximate subdiscontinuity curves in the approximate wavefront solution. In this section we show that their number is uniformly bounded with respect to approximation parameter.

In Section 5, we finally give the proof for Theorem 1.1 and Theorem 1.2.

In Section 6, we construct a strictly hyperbolic 2×2 system of conservation laws, which is not piecewise genuinely nonlinear and its admissible solutions to some initial datum do not have the structural properties described in Theorem 1.1.

2. SOLUTION OF RIEMANN PROBLEM

As in [2], for a fixed point $u^0 \in \Omega$ and $i \in \{1, \dots, N\}$, one can construct smooth vector-value maps $\tilde{r}_i = \tilde{r}_j(u, v_i, \sigma_i)$ for $(u, v_i, \sigma_i) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$ with $\tilde{r}_i(u, 0, \sigma) = r_i(u)$ for all u, σ_i . Setting $l_i^0 := l_i(u^0)$, we can normalize \tilde{r}_i such that

$$(2.1) \quad \langle l_j^0, \tilde{r}_i(u, v_i, \sigma_i) \rangle = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

Writing the speed function $\tilde{\lambda}_i := l_i^0 \cdot Df(u) \tilde{r}_i(u, v_i, \sigma_i)$, we consider the set, for some $\delta_0, C_0 > 0$ fixed and $s > 0$

$$\begin{aligned} \Gamma_i(s, u^-) := & \left\{ \gamma \in \text{Lip}([0, s], \mathbb{R}^{N+2}), \gamma(\xi) = (u(\xi), v_i(\xi), \sigma_i(\xi)) \right. \\ & u(0) = u^-, |u(\tau) - u^-| = \tau, v_i(0) = 0, \\ & \left. |v_i| \leq \delta_1, |\sigma_i(\tau) - \lambda_i(u^0)| \leq 2C_0\delta_1 \right\}. \end{aligned}$$

Given a curve $\gamma \in \Gamma_i$, we define the scalar flux function

$$(2.2) \quad \tilde{f}_i(\tau, \gamma) = \int_0^\tau \tilde{\lambda}_i(u(\xi), v_i(\xi), \sigma_i(\xi)) d\xi.$$

Moreover, we define the lower convex envelope of \tilde{f}_i on $[a, b] \subset [0, s]$ as

$$\begin{aligned} \text{conv}_{[a,b]} \tilde{f}_i(\tau; \gamma) := & \inf \left\{ \theta \tilde{f}_i(\tau', \gamma) + (1 + \theta) \tilde{f}_i(\tau'', \gamma); \right. \\ & \left. \theta \in [0, 1], \tau', \tau'' \in [a, b], \tau = \theta\tau' + (1 + \theta)\tau'' \right\}. \end{aligned}$$

Then define a nonlinear operator $\mathcal{T}_{i,s} : \Gamma_i(s, u^-) \rightarrow \Gamma_i(s, u^-)$ by setting $\gamma = \tilde{\gamma} := (\tilde{u}, \tilde{v}_i, \tilde{\sigma}_i)$, where

$$(2.3) \quad \begin{cases} \tilde{u}(\tau) = u^- + \int_0^\tau \tilde{r}_i(u(\xi), v_i(\xi), \sigma(\xi)) d\xi, \\ \tilde{v}_i(\tau) = \tilde{f}_i(\tau; \gamma) - \text{conv}_{[0,s]} \tilde{f}_i(\tau, \gamma), \\ \tilde{\sigma}_i = \frac{d}{d\tau} \text{conv}_{[0,s]} \tilde{f}_i(\tau; \gamma). \end{cases}$$

One can show that \mathcal{T} is a contraction in $\Gamma_i(s, u^-)$ with respect to the distance

$$D(\gamma, \gamma') := \delta_1 \|u - u'\|_{L^\infty} + \|v_i - v'_i\|_{L^1} + \|v_i \sigma_i - v'_i \sigma'_i\|_{L^1},$$

where

$$\gamma = (u, v_i, \sigma_i), \gamma' = (u', v'_i, \sigma'_i) \in \Gamma_i(s, u^-).$$

Hence, for any s and u^- in a small neighborhood of u^0 , \mathcal{T}_i has a unique fixed point, which is a Lipschitz continuous curve

$$\bar{\gamma}(\tau) = (\bar{u}(\tau; u^-, s), \bar{v}_i(\tau; u^-, s), \bar{\sigma}_i(\tau; u^-, s)), \quad \tau \in [0, s].$$

Then the elementary curve for i -th family is defined as

$$(2.4) \quad T_i[u^-](s) := \bar{u}(s; u^-, s).$$

After adopting the following notations

$$(2.5) \quad \begin{aligned} \sigma_i[u^-](s, \tau) &:= \bar{\sigma}_i(\tau; u^-, s), \\ \tilde{f}_i[u^-](s, \tau) &:= \tilde{f}_i(\tau; \bar{\gamma}). \end{aligned}$$

and recalling that the Riemann problem is the Cauchy problem (1.1) with piecewise constant initial data of the form

$$(2.6) \quad u_0(x) = \begin{cases} u^L & x < 0, \\ u^R & x > 0, \end{cases}$$

where u^L, u^R are two constants. One has the following theorem [3].

Theorem 2.1. *For every $u \in \Omega$ and $s > 0$ sufficiently small,*

- (1) *N Lipschitz continuous curves $s \mapsto T_i[u](s) \in \Omega$, $i = 1, \dots, N$, satisfying $\lim_{s \rightarrow 0} \frac{d}{ds} T_i[u](s) = r_i(u)$,*
- (2) *N Lipschitz continuous functions $(s, \tau) \mapsto \sigma_i[u](s, \tau)$, with $0 \leq \tau \leq s$ and $i = 1, \dots, N$, satisfying $\tau \mapsto \sigma_i[u](s, \tau)$ which are increasing and such that $\sigma_i[u](s, 0) = \lambda_i(u)$,*

with the following properties.

When $u^L \in \Omega$, $u^R = T_i[u^L](s)$, for some s sufficiently small, the unique Liu admissible solution of the Riemann problem (1.1)-(2.6) is defined a.e. by

$$(2.7) \quad u(x, t) := \begin{cases} u^L & x/t < \sigma_i[u^L](s, 0), \\ T_i[u^L](\tau) & x/t = \sigma_i[u^L](s, \tau), \tau \in [0, s], \\ u^R & x/t > \sigma_i[u^L](s, s). \end{cases}$$

For the case when $s < 0$, a right state $u^R = T_i[u^L](s)$ can be constructed in the same way as before, except that one replaces $\text{conv}_{[0, s]} \tilde{f}_i$ in (2.3) with the upper concave envelope of \tilde{f}_i on $[s, 0]$:

$$\begin{aligned} \text{conc}_{[a, b]} \tilde{f}_i(\tau; \gamma) &:= \sup \left\{ \theta f_i(\tau', \gamma) + (1 + \theta) f_i(\tau'', \gamma); \right. \\ &\quad \left. \theta \in [0, 1], \tau', \tau'' \in [a, b], \tau = \theta \tau' + (1 + \theta) \tau'' \right\}, \end{aligned}$$

and looks at the fixed point of the integral system (2.3) on the interval $[s, 0]$.

Because of the assumption (2.1) and the definition (2.4), the elementary curve $T_i[u^L]$ is parameterized by its i -th component relative to the basis $r_1(u^0), \dots, r_N(u^0)$ i.e.

$$(2.8) \quad s = \langle l_i^0, T_i[u^L](s) - u^L \rangle.$$

Remark 2.2. In [3], it is proved that if $u^L, u^R \in \Delta_i^k$ with some k even (odd) and $u^R = T_i[u^L](s)$, $s > 0$ ($s < 0$), the solution u of the Riemann problem with the initial data (2.6) is a *center rarefaction wave*, that is for $t > 0$,

$$u(x, t) = \begin{cases} u^L & \text{if } x/t < \lambda_i(u^L), \\ u^R & \text{if } x/t > \lambda_i(u^R), \\ R_i[u^L](\tau) & \text{if } x/t \in [\lambda_i(u^L), \lambda_i(u^R)], \ x/t = \lambda_i(R_i[u^L](\tau)), \end{cases}$$

where $\tau \in [0, s]$ ($\tau \in [s, 0]$) such that $s = \langle l_i^0, R_i[u^L](s) - u^L \rangle$. Notice that u is smooth for $t > 0$.

Remark 2.3. As shown in [3] (also see Remark 4 in [1] and Section 4 of [10]), under the assumption of piecewise genuine nonlinearity, the solution of the Riemann problem provided by (2.7) is a composed wave of the i -th family containing a finite number of rarefaction waves and Liu admissible discontinuities. Recalling Theorem 2.1, one knows that the regions where the v_i -component of the solution to (2.3) vanishes correspond to rarefaction waves, while the regions where the v_i -component of the solution to (2.3) is different from zero correspond to admissible discontinuities.

The Liu admissible solution [3] of a Riemann problem for (1.1)-(2.6) is obtained by constructing a Lipschitz continuous map

$$(2.9) \quad \mathbf{s} := (s_1, \dots, s_N) \mapsto T[u^L](\mathbf{s}) := T_N[T_{N-1}[\dots[T_1[u^L](s_1)]\dots](s_{N-1})(s_N) = u^R,$$

which is one to one from a neighborhood of the origin onto a neighborhood of u^L . Then we can uniquely determine intermediate states $u^L = \omega_0, \omega_1, \dots, \omega_N = u^R$, and the *wave strength* s_1, s_2, \dots, s_N such that

$$\omega_i = T_i[\omega_{i-1}](s_i), \quad i = 1, \dots, N,$$

provided that $|u^L - u^R|$ is sufficiently small.

By Theorem 2.1, each Riemann problem with initial data

$$(2.10) \quad u_0 = \begin{cases} \omega_{i-1} & x < 0, \\ \omega_i & x > 0, \end{cases}$$

admits a self-similar solution u_i , containing only i -waves. We call u_i the i -th *elementary composite wave* or simply *i -wave*. Therefore, under the strict hyperbolicity assumption, the solution of the Riemann problem with the initial data (2.6) is obtained by piecing together the self-similar solutions of the Riemann problems given by (1.1)-(2.10).

Indeed, from the strict hyperbolicity assumption (1.2), the speed of each elementary i -th wave in the solution u_i is inside the interval $[\check{\lambda}_{i-1}, \check{\lambda}_i]$ if $s \ll 1$, so that the solution of the general Riemann problem (1.1)-(2.6) is then given by

$$(2.11) \quad u(x, t) = \begin{cases} u^L & x/t < \check{\lambda}_0, \\ u_i(x, t) & \check{\lambda}_{i-1} < x/t < \check{\lambda}_i, i = 1, \dots, N, \\ u^R & x/t > \check{\lambda}_N. \end{cases}$$

3. DESCRIPTION OF WAVE-FRONT TRACKING APPROXIMATION

In [1], the authors provide an algorithm of wave-front tracking approximation for vanishing viscosity BV solutions to the strictly hyperbolic system which is much more general than the case discussed here. We modify a little about its algorithm in order to simplify our analysis. Due to Theorem 2.1, one knows that the solution constructed by such approximation is Liu admissible.

Wave-front tracking approximation is an algorithm which produces piecewise constant approximate solutions to the Cauchy problem (1.1). In order to construct approximate wave-front tracking solutions, given a fixed $\epsilon > 0$, we first choose a piecewise constant function u_0^ϵ which is a good approximation to initial data u_0 such that

$$(3.1) \quad \text{Tot.Var.}\{u_0^\epsilon\} \leq \text{Tot.Var.}\{u_0\}, \quad \|u_0^\epsilon - u_0\|_{L^1} < \epsilon,$$

and u_0^ϵ only has finite jumps. Let $x_1 < \dots < x_m$ be the jump points of u_0^ϵ . For each $\alpha = 1, \dots, m$, we approximately solve the Riemann problem (just shifting the center from $(0, 0)$ to $(0, x_\alpha)$) with the initial data of the jump $[u_0^\epsilon(x_\alpha-), u_0^\epsilon(x_\alpha+)]$ by a function $w(x, t) = \phi(\frac{x-x_\alpha}{t})$ where ϕ is a piecewise constant function. The straight lines where the discontinuities locate are called *wave-fronts* (or just *fronts* for short). The wave-fronts can prolong until they interact with other fronts, then at the interaction point, the corresponding Riemann problem is approximately solved and several new fronts are generated forward. Then one tracks the wave-fronts until they interact with other wave-fronts, etc... In order to avoid the algorithm to produce infinite many wave-fronts in finite time, different kinds of approximate Riemann solvers should be introduced.

3.0.1. The approximate i -th elementary wave. Suppose that u_i is an i -th elementary composite wave which is obtain by solving Riemann problem with initial data (2.10) where $\omega_i = T_i[\omega_{i-1}](s_i)$. For notational convenience, we write $\sigma_i(\tau) := \sigma_i[\omega_{i-1}](s_i, \tau)$. Let

$$p := \left\lceil \frac{\sigma_i(s_i) - \sigma_i(0)}{\epsilon} \right\rceil + 1$$

and

$$\vartheta_l := \sigma_i(0) + \frac{l}{p} [\sigma_i(s_i) - \sigma_i(0)], \quad l = 0, \dots, p-1.$$

We set

$$\omega_{i-1,l} = T_i[\omega_{i-1}](s_{i,l}),$$

where

$$s_{i,l} := \begin{cases} \min\{s \in [0, s_i], \sigma_i(s) = \vartheta_l\}, & s_i \geq 0, \\ \max\{s \in [s_i, 0], \sigma_i(s) = \vartheta_l\}, & s_i \leq 0. \end{cases}$$

Then the i -th elementary composite wave u_i is approximated by \tilde{u}_i as the following,

$$(3.2) \quad \tilde{u}_i(x, t) = \begin{cases} \omega_{i-1} & x/t < \vartheta_{i,0}, \\ \omega_{i-1,l}, & \vartheta_{i-1,l-1} < x/t < \vartheta_{i,l}, \quad (l = 1, \dots, p-1), \\ \omega_i & x/t > \vartheta_{i,p-1}. \end{cases}$$

Notice that \tilde{u}_i consists of p fronts with small strength.

3.0.2. Approximate Riemann solver. Suppose at the point (t_1, x_1) , a wave-front $[u^L, u^M]$ of strength s' belonging to i' -th family interacts from the left with a wave-front $[u^M, u^R]$ of strength s'' belonging to i'' -th family for some $i', i'' \in \{1, \dots, N\}$ such that

$$u^M = T_{i'}[u^L](s'), \quad u^R = T_{i''}[u^M](s'').$$

Assume that $|u^L - u^R|$ sufficiently small. Then at the interaction point, the Riemann problem with the initial data of the jump $[u^L, u^R]$ may be solved by two kinds approximate Riemann solver according to different situation.

- *Accurate Riemann solver:* It replaces each elementary composite wave of the exact Riemann solution (refers to u_i in the solution (2.11)) with an approximate i -th elementary wave defined by (3.2).
- *Simplified Riemann solver:* It only generates the approximate elementary waves belong to i' -th and i'' -th families with the corresponding strength s' and s'' as the incoming ones if $i' \neq i''$ or the approximate i' -th elementary waves of strength $s' + s''$ if $i' = i''$. The simplified Riemann solver collects the remaining new waves into a single *nonphysical front*, traveling with a constant speed $\hat{\lambda}$, strictly larger than all characteristic speeds. Therefore, usually the simplified Riemann solver generates less outgoing fronts after an interaction than the accurate Riemann solver.

Since the simplified Riemann solver produces nonphysical wave-fronts and they can not interact with each other, one needs an approximate Riemann solver defined for the interaction between, for example, a physical front of the i -th family with strength s , connecting u^M, u^R and a nonphysical front (coming from the left) connecting the left value u^L and u^M traveling with speed $\hat{\lambda}$.

- *Crude Riemann solver:* It generates an approximate i -th elementary wave connecting u^L and $\tilde{u}^M = T_i[u^L](s)$ and a nonphysical wave-front joining \tilde{u}^M and u^R , traveling with speed $\hat{\lambda}$. In the following, for simplicity, we just say that the non-physical fronts belong to the $(N+1)$ -th characteristic field.

Remark 3.1. It is not restrictive to assume that at each time $t > 0$, at most one interaction takes place, involving exactly two incoming fronts, because one can always slightly change the speeds of the incoming fronts if more than two fronts meet at the same point. It is sufficient to require that the error vanishes when the approximation solutions converge to the exact solution. Actually, suppose $x=y(t)$ is a front in an approximate solution u with parameter ϵ and $u^L = u(t, x-)$ and $u^R = u(t, x+)$ such that

$$(3.3) \quad u^R = T_i[u^L](s)$$

for some index $i \in \{1, \dots, N\}$ and wave strength s . Then the following holds

$$(3.4) \quad |\dot{y}(t) - \sigma_i[u^L](s, \tau)| \leq 2\epsilon, \quad \forall \tau \in [0, s].$$

Remark 3.2. There are three kinds of physical wave-fronts. Suppose $[u^L, u^R]$ is wave front in an approximate solution, and $u^R = T_i[u^L](s)$ ($s > 0$). Recalling the notation (2.5) and Remark 2.3, and writing $\tilde{f}_i(\tau) := \tilde{f}_i[u^L](s, \tau)$, one knows that $[u^L, u^R]$ may be one of the following three kinds of fronts:

- *Discontinuity front* if $\tilde{f}_i(\tau) > \text{conv}_{[0,s]} \tilde{f}_i(\tau)$, $\forall \tau \in [0, s] \setminus \mathcal{S}$, where \mathcal{S} is set of finite cardinality.

- *Rarefaction front* if $\tilde{f}_i(\tau) = \text{conv}_{]0,s[} \tilde{f}_i(\tau)$, $\forall \tau \in]0,s[$. In this case, $u^L, u^R \in \Delta_i^k$ for some k even and $u^R = R_i[u^L](s)$.
- *Mixed front* if there exist $]a,b[\subset]0,s[$ such that $\tilde{f}_i(\tau) > \text{conv}_{]0,s[} \tilde{f}_i(\tau)$, $\forall \tau \in]a,b[$ and $]c,d[\subset]0,s[$ such that $\tilde{f}_i(\tau) = \text{conv}_{]0,s[} \tilde{f}_i(\tau)$, $\forall \tau \in]c,d[$.

3.0.3. *Interaction potential and BV estimates.* In order to check the total variations of approximate solutions are uniformly bounded with respect to time, one needs the estimate on the difference between the strength of the incoming waves and the strength of the outgoing waves produced by an interaction. Suppose two wave-fronts with strength s' and s'' interact and $\tilde{f}'_i, \tilde{f}''_i$ are the corresponding scalar flux function defined by (2.2). We define the *amount of interaction* $\mathcal{I}(s', s'')$ between s' and s'' .

When s' and s'' belong to different characteristic families, set

$$(3.5) \quad \mathcal{I}(s', s'') = |s' s''|.$$

When s', s'' belong to the same family,

- (a) If $s'' > 0$, we set

$$\begin{aligned} \mathcal{I}(s', s'') := & \int_0^{s'} \left| \text{conv}_{[0,s']} \tilde{f}'_i(\xi) - \text{conv}_{[0,s'+s'']} (\tilde{f}'_i \cup \tilde{f}''_i)(\xi) \right| d\xi \\ & + \int_{s'}^{s'+s''} \left| \text{conv}_{[0,s']} (\tilde{f}'(s') \tilde{f}''_i(\xi - s')) - \text{conv}_{[0,s'+s'']} (\tilde{f}'_i \cup \tilde{f}''_i)(\xi) \right| d\xi, \end{aligned}$$

- (b) if $-s' \leq s'' < 0$, we set

$$\begin{aligned} \mathcal{I}(s', s'') := & \int_0^{s'+s''} \left| \text{conv}_{[0,s']} \tilde{f}'_i(\xi) - \text{conv}_{[0,s'+s'']} \tilde{f}'_i(\xi) \right| d\xi \\ & + \int_{s'+s''}^{s'} \left| \text{conv}_{[0,s']} \tilde{f}'(\xi) - \text{conc}_{[s'+s'',s']} \tilde{f}'_i(\xi) \right| d\xi, \end{aligned}$$

- (c) if $s'' < -s'$, we set

$$\begin{aligned} \mathcal{I}(s', s'') := & \int_{s''}^{-s'} \left| \text{conc}_{[s'',0]} \tilde{f}''_i(\xi) - \text{conc}_{[s'',-s']} \tilde{f}''_i(\xi) \right| d\xi \\ & + \int_{-s'}^0 \left| \text{conv}_{[s'',0]} (\tilde{f}'(\xi)) - \text{conv}_{[-s',0]} \tilde{f}'_i(\xi) \right| d\xi, \end{aligned}$$

Throughout the paper, we write $A \lesssim B$ ($A \gtrsim B$) if there exists a constant $C > 0$ which only depends on the system (1.1) such that $A \leq CB$ ($A \geq CB$).

Recall the Lipschitz continuous map T defined in (2.9) and suppose $u^M = T_i[u^L](s_1)$, $u^R = T_j[u^M](s_2)$ and $u^R = T[u^L](s)$. By Glimm's interaction estimates proved in [2] (also see Lemma 1 in [1]), one has

$$(3.6) \quad |s - s_1 - s_2| \lesssim \mathcal{I}(s_1, s_2)$$

where $\mathbf{s}_h = (s_1, s_2, \dots, s_N)$, $h = 1, 2$ with $s_n = 0$ for $n \neq h$.

At each time $t > 0$ when no interaction occurs, and the approximate solution u has jumps at x_1, \dots, x_m , we denote by

$$\omega_1, \dots, \omega_m, \quad s_1, \dots, s_m, \quad i_1, \dots, i_m,$$

their left states, signed strengths and characteristic families respectively: the sign of s_α is given by the respective orientation of $dT_i[u](s)/ds$ and r_i , if the jump at x_α belongs to the i -th family. The total variation of u_ν will be computed as

$$V(t) := \sum_{\alpha} |s_\alpha|.$$

Since $T_i[u^0]$ is a Lipschitz continuous function and the Lipschitz constant is uniformly bounded for any $u^0 \in \Omega$, one has

$$(3.7) \quad \text{Tot.Var}\{u(\cdot, t)\} \lesssim V(t).$$

Then the estimate of increasing of $\text{Tot.Var}\{u_\nu(\cdot, t)\}$ turns out to be the estimate of the total amount of interaction. Following [2], we define the *Glimm wave interaction potential* as follows:

$$\mathcal{Q}(t) := \sum_{\substack{i_\alpha > i_\beta \\ x_\alpha < x_\beta}} |s_\alpha s_\beta| + \frac{1}{4} \sum_{i_\alpha = i_\beta < N+1} \int_0^{|s_\alpha|} \int_0^{|s_\beta|} |\sigma_{i_\beta}[\omega_\beta](s_\beta, \tau'') - \sigma_{i_\alpha}[\omega_\alpha](s_\alpha, \tau')| d\tau' d\tau''.$$

Denoting the time jumps of the total variation and the Glimm potential as

$$\Delta V(\tau) = V(\tau+) - V(\tau-), \quad \Delta \mathcal{Q}(\tau) = \mathcal{Q}(\tau+) - \mathcal{Q}(\tau-),$$

the fundamental estimates are the following (Lemma 5 in [1]): in fact, when two wave-fronts with strength s' , s'' interact,

$$(3.8a) \quad \Delta \mathcal{Q}(\tau) \lesssim \mathcal{I}(s', s''),$$

$$(3.8b) \quad \Delta V(\tau) \lesssim \mathcal{I}(s', s'').$$

Thus one defines the *Glimm functional*

$$\Upsilon(t) := V(t) + C_0 \mathcal{Q}(t)$$

with C_0 suitable constant, so that Υ decreases at any interaction. Using this functional, one can prove that their total variations are uniformly bounded. Moreover, one can also show that the number of wave-fronts remains finite for all time (see section 6.1 of [1]). This makes sense for the construction of approximate wave-front tracking solutions.

3.0.4. Construction of the approximate solutions and their convergence to exact solution. The construction starts at initial time $t = 0$ with a given $\epsilon > 0$, by taking $u_{0,\epsilon}$ as a suitable piecewise constant approximation of initial data u_0 , satisfying (3.1). At the jump points of $u_{0,\epsilon}$, we locally solve the Riemann problem by accurate Riemann solver. The approximate solution then can be prolonged until a first time t_1 when two wave-fronts interact. Again we solve the Riemann problem at the interaction point by an approximate Riemann solver. Whenever the amount of interaction (see Section 3.0.3 for the definition) of the incoming waves is larger than some threshold parameter $\rho = \rho(\epsilon) > 0$, we shall adopt the accurate Riemann solver. Instead, in the case where the amount of interaction of the incoming waves is less than ρ , we shall adopt the simplified Riemann solvers. The threshold ρ is suitably chosen so that the number of wave-fronts remains finite for all times. And we will apply the crude Riemann solver if one of the incoming wave-front is non-physical front. One can show that the number of wave fronts is uniformly bounded (see Section 6.2 in [1]).

We call such approximate solutions *ϵ -approximate front tracking solutions*. At each time t when there is no interaction, the restriction $u_\epsilon(t)$ is a step function whose jumps are located along straight lines in the (x, t) -plane.

Let $\{\epsilon_\nu\}_{\nu=1}^\infty$ be a sequence of positive real numbers converging to zero. Consider a corresponding sequence of ϵ_ν -approximate front tracking solutions $u_\nu := u_{\epsilon_\nu}$ of (1.1): it is standard to show that the functions $t \mapsto u_\nu(t, \cdot)$ are uniformly Lipschitz continuous in L^1 norm. since (3.7) and (3.8) hold independent of the parameter ϵ_ν , $u_\nu(t, \cdot)$ have uniformly bounded total variation. Therefore by Helly's theorem, u_ν converges up to a subsequence in $\mathbb{L}_{\text{loc}}^1(R^+ \times \mathbb{R})$ to some function u , which is a weak solution of (1.1).

It can be shown that by the choice of the Riemann solver in Theorem 2.1, the solution obtained by the front tracking approximation coincides with the unique vanishing viscosity solution [3]. Furthermore, there exists a closed domain $\mathcal{D} \subset L^1(\mathbb{R}, \Omega)$ and a unique distributional solution u , which is a Lipschitz semigroup $\mathcal{D} \times [0, +\infty[\rightarrow \mathcal{D}$ and which for piecewise constant initial data coincides, for a small time, with the solution of the Cauchy problem obtained piecing together the standard entropy solutions of the Riemann problems. Moreover, it lives in the space of BV functions.

For simplicity, the pointwise value of u is its L^1 representative such that the restriction map $t \mapsto u(t)$ is continuous from the right in L^1 and $x \mapsto u(x, t)$ is right continuous from the right.

3.0.5. *Further estimates.* To each u_ν , we define the *measure* μ_ν^I of *interaction* and the *measure* μ_ν^{IC} of *interaction and cancelation* concentrated on the set of interaction points as follows. If two physical fronts belonging to the families $i', i'' \in \{1, \dots, N\}$ with strength s', s'' interact at point P , we denote

$$(3.9a) \quad \mu_\nu^I(\{P\}) := \mathcal{I}(s', s''),$$

$$(3.9b) \quad \mu_\nu^{IC}(\{P\}) := \mathcal{I}(s', s'') + \begin{cases} |s'| + |s''| - |s' + s''| & i' = i'', \\ 0 & i' \neq i''. \end{cases}$$

The wave strength estimates (3.6) yields balance principles for the wave strength of approximate solutions. More precisely, given a polygonal region Γ with edges transversal to the waves it encounters. Denote by $W_{\nu, \text{in}}^{i\pm}, W_{\nu, \text{out}}^{i\pm}$ the positive (+) or negative (−) i -waves in u^ν entering or exiting Γ , and let $W_{\nu, \text{in}}^i = W_{\nu, \text{in}}^{i+} - W_{\nu, \text{in}}^{i-}$, $W_{\nu, \text{out}}^i = W_{\nu, \text{out}}^{i+} - W_{\nu, \text{out}}^{i-}$. Then the measure of interaction and the measure of interaction-cancelation control the difference between the amount of exiting i -waves and the amount of entering i -waves w.r.t. the region as follows:

$$|W_{\nu, \text{out}}^i - W_{\nu, \text{in}}^i| \lesssim \mu_\nu^I(\Gamma),$$

$$|W_{\nu, \text{out}}^{i\pm} - W_{\nu, \text{in}}^{i\pm}| \lesssim \mu_\nu^{IC}(\Gamma).$$

The above estimates are fairly easy consequences of the interaction estimates (3.8) and the definition of μ_ν^I, μ_ν^{IC} . On the other hand, the uniform boundedness of $\text{Tot.Var.}\{u(\cdot, t)\}$ w.r.t. time t and parameter ν implies that μ_ν^I and $\mu_\nu^{IC} - \nu$ are bounded measures for all ν .

By taking a subsequence and using the weak compactness of bounded measures, there exist bounded measures μ^I and μ^{IC} on $\mathbb{R}^+ \times \mathbb{R}$ such that the following weak convergence holds:

$$\mu_\nu^I \rightharpoonup \mu^I, \quad \mu_\nu^{IC} \rightharpoonup \mu^{IC}.$$

4. CONSTRUCTION OF SUBDISCONTINUITY CURVES

Suppose $[u^L, u^R]$ with $u^R = T_i[u^L](s)$ is a wave front of i -th family in the approximate solution u_ν , and the wave curve $\tau \mapsto T_i[u^L](\tau)$ (see Theorem 2.1) intersects Z_i^j, \dots, Z_i^{j+p} at u_j, \dots, u_{j+p} for $0 \leq \tau \leq s$, such that

$$u_j = T_i[u^L](s_j).$$

Then we say that the wave front $[u^L, u^R]$ has (i, k) -substrength $s_i^k := s_{k+1} - s_k$ and we decompose the front into (i, k) -subdiscontinuity fronts with strength s_i^k , where $k \in \{j, \dots, j+p-1\} \cap 2\mathbb{Z}$ when $s > 0$, or $k \in \{j, \dots, j+p-1\} \cap (2\mathbb{Z} + 1)$ when $s < 0$. The points u_k and u_{k+1} are connected by the part of the curve $T_i[u^L](\cdot)$ inside Δ_i^k .

We denote the family of all (i, k) -subdiscontinuity fronts as \mathcal{S}_i^k .

It is obviously that only mixed fronts and discontinuity fronts can have (i, k) -substrength $s_i^k > 0$ for some k , which means they can be decomposed into subdiscontinuity fronts.

Lemma 4.1. *In a wave-front tracking approximation solution, an interaction can only generate at most one subdiscontinuity front with strength s_i^k for some i, k .*

Proof. By the construction of approximate Riemann solver and the uniformly small total variation of approximate solutions, it is sufficient to prove that the Lipschitz continuous curve $T_i[u^0](\cdot) : [0, s] \rightarrow \mathbb{R}^N$ can intersect with Z_i^j at most once for any $u^0 \in \Omega$ and all j if $s > 0$ is sufficiently small. In fact, by Theorem 2.1 $\lim_{t \rightarrow 0} T_i[u^0](t) = r_i(u^0)$, one has for $t \in [0, s]$,

$$\left| \frac{d}{dt} T_i[u^0](t) - r_i(T_i[u^0](t)) \right| \lesssim t.$$

Recall the assumption $r_i(u) \cdot \mathbf{n} > 0$ on Z_i^j , one has

$$\frac{d}{dt} T_i[u^0](t) \cdot \mathbf{n} > 0 \text{ on } Z_i^j \quad \text{for } t \in [0, s]$$

as long as s small enough. This concludes that $T_i[u^0](\cdot) : [0, s] \rightarrow \mathbb{R}^N$ can intersect Z_i^j at most once. \square

Suppose $y_1 \in \mathcal{S}_i^{k'}$, $y_2 \in \mathcal{S}_i^{k''}$ and y_1, y_2 belong to the same wave front, then we say artificially y_1 is on the left (right) of y_2 if k', k'' are even (odd) and $k' < k''$ ($k' > k''$). Then it is easy to see the following lemma.

Lemma 4.2. *Suppose two subdiscontinuity fronts $y_1 \in \mathcal{S}_i^{k'}$, $y_2 \in \mathcal{S}_i^{k''}$ interact and generate two subdiscontinuity fronts $y'_1 \in \mathcal{S}_i^{k'}$, $y'_2 \in \mathcal{S}_i^{k''}$. Then if $k' < k''$ ($k' > k''$), y_1 must be on the left (right) of y_2 and also y'_1 must be on the left (right) of y'_2 .*

Base on these two lemmas, we can follow the idea of [7] to define approximate subdiscontinuity curves.

Definition 4.3. Given $\epsilon \neq 0$, in u_ν , an (ϵ, i, k) -approximate subdiscontinuity curve is a polygonal line in (x, t) -plane with nodes $(t_0, x_0), (t_1, x_1), \dots, (t_n, x_n)$ satisfying

- (1) (t_j, x_j) are interaction points with $0 \leq t_0 < t_1 < \dots < t_n$.
- (2) For $1 \leq j \leq n$ the segment joining $(t_{j-1}, x_{j-1}), (t_j, x_j)$ is an (i, k) -subdiscontinuity front with $s_i^k \geq \epsilon/2$ when $\epsilon > 0$ ($s_i^k \leq \epsilon/2$ when $\epsilon < 0$), and there is at least one index $j' \in \{1, \dots, n\}$ such that the wave front connecting $(t_{j'-1}, x_{j'-1}), (t_{j'}, x_{j'})$ has (i, k) -substrength $s_i^k \geq \epsilon$ if $\epsilon > 0$ ($s_i^k \leq \epsilon$ if $\epsilon < 0$).
- (3) $\forall k < N$, one selects the (i, k) -subdiscontinuity fronts with larger speed, that is, if two fronts with substrengths $s_i^k > \epsilon/2$ interact at the node (x_k, t_k) , then the front of the (ϵ, i, k) -approximate subdiscontinuity curve is the one coming from the left.

An (ϵ, i, k) -approximate subdiscontinuity curve which is maximal w.r.t. set inclusion is called a *maximal (ϵ, i, k) -approximate subdiscontinuity curve*.

Let $M_{i,\nu}^k(\epsilon)$ be the number of maximal (ϵ, i, k) -approximate subdiscontinuity curves in u_ν .

Lemma 4.4. *For fixed k and ϵ , $M_{i,\nu}^k(\epsilon)$ is uniformly bounded w.r.t ν .*

Proof. We consider the case for $\epsilon > 0$ (and the case when $\epsilon < 0$ is similar). Since the total variation of $u_\nu(0, \cdot)$ is uniformly bounded by (3.1), the number of maximal (ϵ, i, k) -approximate subdiscontinuity curves which start at time $t = 0$ is clearly of order ϵ^{-1} . The number of (ϵ, i, k) -approximate subdiscontinuity curves which start at a time $t_0 > 0$ and do not end in finite time is also of order ϵ^{-1} , because the total variation of $u_\nu(\cdot, t)$ is uniformly bounded w.r.t time t and ν .

Now considering an (ϵ, i, k) -approximate subdiscontinuity curve y_ν which starts at a time $t_0 > 0$ and ends in finite time, we claim that

$$(4.1) \quad \mu_\nu^{IC}(y_\nu) \gtrsim \epsilon^2.$$

If the claim is true, as the total amount of interaction and cancelation in the solution u_ν is uniformly bounded, the number of such (ϵ, i, k) -approximate subdiscontinuity curves is of order ϵ^{-2} .

Thus, combining these situations, we finally obtain the estimate

$$M_{i,\nu}^k(\epsilon) \lesssim \epsilon^{-2},$$

which is uniformly valid as $\nu \rightarrow \infty$.

Now we prove the claim. Suppose one (i, k) -subdiscontinuity front on y_ν with (i, k) -substrength $\alpha > 0$ interact with a front of j -th family with strength β^* at point P , generating an i -wave with (i, k) -substrength $\gamma \geq 0$ which means that either there is a front of i -wave with (i, k) -substrength $\gamma > 0$ or there is no fronts of i -wave having (i, k) -substrength. We also assume that $\gamma < \alpha$. Setting $\theta = \alpha - \gamma$.

First, we consider the case when $i \neq j$, we assume that $i > j$, the case $i < j$ is similar to prove. For notational convenience, we also denote α, β^*, γ as the front themselves.

Assume that α locates on the front $[u^L, u^M]$ with $u^M = T_i[u^L](\alpha^*)$ and $u^R = T_j[u^M](\beta^*)$, γ locates on the front of the i -wave $[\tilde{u}^L, \tilde{u}^R]$ with $\tilde{u}^R = T_i[\tilde{u}^L](\gamma^*)$.

We know that (see section 9.9 of [9])

$$(4.2a) \quad \tilde{u}^L = u^L + s_j \sum_{j < i} r_j(u^M) + O(|s^L|)\alpha^* + o(|s^L|),$$

$$(4.2b) \quad \tilde{u}^R = u^R - s_j \sum_{j > i} r_j(u^M) + O(|s^R|)\beta^* + o(|s^R|),$$

where

$$s^L = (s_1, \dots, s_{i-1}, 0, \dots, 0) \quad \text{and} \quad s^R = (0, \dots, 0, s_{i+1}, \dots, s_N).$$

Then, by (3.6), the assumption $i > j$ and $|u^M - u^R| \lesssim |\beta^*|$, we have

$$(4.3a) \quad |\tilde{u}^L - u^L| \lesssim |\beta^*| + \mathcal{I}(\alpha^*, \beta^*),$$

$$(4.3b) \quad |u^M - \tilde{u}^R| \lesssim |\beta^*| + \mathcal{I}(\alpha^*, \beta^*).$$

From Glimm's interaction estimates (3.6), the parametrization (2.8) and the estimates (4.3), one concludes that the difference of (i, k) -substrength between α and γ is controlled by the amount of interaction and the strength of the wave β^* , that is

$$(4.4) \quad \theta = \alpha - \gamma \lesssim \mathcal{I}(\alpha^*, \beta^*) + |\beta^*|.$$

From definition of \mathcal{I} (see Section 3.0.3), we know that here $\mathcal{I}(\alpha^*, \beta^*) = |\alpha^* \beta^*|$. So we get $|\beta^*| \gtrsim \theta$ since $|\alpha^*| \ll 1$. Therefore from $|\alpha^*| \geq \epsilon/2$, by the estimates in (i)-(iii), we obtain

$$\mathcal{I}(\alpha^*, \beta^*) \gtrsim \epsilon \theta.$$

By notation of interaction measure (3.9a), one obtains

$$(4.5) \quad \mu_\nu^I(P) \gtrsim \epsilon \theta.$$

Next we consider the case when $i = j$ and α is on the left of β^* . (It is similar for the case when α is on the right of β^* .)

First we assume that $\alpha > 0, \beta^* < 0$. Since the decreasing of the (i, k) -substrength can only be caused by interaction and cancellation effects, then similarly one has

$$\theta \lesssim |\beta^*| + \mathcal{I}(\alpha^*, \beta^*).$$

Second, we assume that $\alpha > 0, \beta^* > 0$.

From the equality (4.2a), one know that

$$(4.6) \quad \langle l^0, \tilde{u}^L - u^L \rangle \lesssim \mathcal{I}(\alpha^*, \beta^*).$$

From the equality (4.2b),

$$\begin{aligned} & \langle l^0, u^M - \tilde{u}^R \rangle + \langle l^0, u^M - u^R \rangle \\ &= \langle l^0, u^R - \tilde{u}^R \rangle \lesssim \mathcal{I}(\alpha^*, \beta^*). \end{aligned}$$

Since $\langle l^0, u^M - u^R \rangle = -\beta^* < 0$, one has

$$(4.7) \quad \langle l^0, u^M - \tilde{u}^R \rangle \lesssim \mathcal{I}(\alpha^*, \beta^*).$$

Noticing that (4.6) and (4.7) implies that the decreasing of the (i, k) -substrength can be controlled by the amount of interaction, one has

$$\theta \lesssim \mathcal{I}(\alpha^*, \beta^*).$$

Therefore from the definition of μ_ν^{IC} (3.9b), in the case when $i = j$ one has

$$(4.8) \quad \mu_\nu^{IC}(P) \gtrsim \theta.$$

From (2) of Definition (4.3), the (i, k) -substrength all front outgoing from the terminal point of y_ν must be less than $\epsilon/2$ and there is at least one front on y_ν has (i, k) -substrength larger than ϵ . Then by (4.5) and (4.8), one can conclude that the claim (4.1) is true. \square

Up to a subsequence, one can assume that $M_{i,\nu}^k(\epsilon) = \bar{M}_i^k(\epsilon)$ is a constant independent of ν . Denote

$$y_{m,\nu}^{k,\epsilon} : [t_{m,\nu}^{k,\epsilon-}, t_{m,\nu}^{k,\epsilon+}] \rightarrow \mathbb{R}, \quad m = 1, \dots, \bar{M}_i^k(\epsilon)$$

as the maximal (ϵ, i, k) -approximate subdiscontinuity curves in u_ν .

Define $\mathcal{T}_{i,\nu}^k(\epsilon)$ as the collection of all maximal (ϵ, i, k) -approximate subdiscontinuity curves in u_ν for fixed ϵ, i and k , i.e.

$$\mathcal{T}_{i,\nu}^k(\epsilon) = \{y_{m,\nu}^{k,\epsilon} : m = 1, \dots, \bar{M}_i^k(\epsilon)\}.$$

Set

$$\mathcal{T}_{i,\nu} := \bigcup_{k,\epsilon} \mathcal{T}_{i,\nu}^k(\epsilon).$$

as the family of all approximate subdiscontinuity curves of i -th family in u_ν .

Up to a diagonal argument and by a suitable labeling of the curves, one can assume that for each fixed h, m , as $\nu \rightarrow \infty$ the Lipschitz continuous curves $y_{m,\nu}^{k,\epsilon}$ converge uniformly to some Lipschitz continuous curves $y_m^{k,\epsilon}$ which are called (ϵ, i, k) -subdiscontinuity curves. Let us denote by

$$\mathcal{T}_i^k(\epsilon) := \{y_m^{k,\epsilon} : m = 1, \dots, \bar{M}_i^k(\epsilon)\}$$

the collection of all these limiting curves for fixed i, k, ϵ .

Let

$$\mathcal{T}_i := \bigcup_{k,\epsilon} \mathcal{T}_i^k(\epsilon).$$

denote the collection of all these i -subdiscontinuity curves.

Lemma 4.5. *Let $y_m^k :]t_m^-, t_m^+[\rightarrow \mathbb{R}$ be an (ϵ, i, k) -subdiscontinuity curve. If $t \in]t_m^-, t_m^+[$ is such that $(t, y_m^k(t)) \notin \Theta$, then the derivative $\dot{y}_m^k(t)$ exists.*

Proof. There exist $\epsilon_0 > 0$ and $y_{m,\nu}^k \in \mathcal{T}_{i,\nu}^k(\epsilon_0)$, such that

$$y_{m,\nu}^k \rightarrow y_m^k,$$

as $\nu \rightarrow \infty$ and the substrength of $y_{m,\nu}^k$ is $|s_i^k| \geq \epsilon_0$.

Since in the wave-front tracking approximation, the change of speed of i -subdiscontinuity fronts is controlled by the measure μ_ν^{IC} .

Then for any $\delta_\nu \rightarrow 0$, we deduce that

$$\limsup_{\nu \rightarrow \infty} \sup_{|t-t'| < \delta_\nu} |\dot{y}_{m,\nu}^k(t) - \dot{y}_{m,\nu}^k(t')| = 0.$$

From uniformly convergence $y_{m,\nu}^k \rightarrow y_m^k$ on a neighborhood of t , we obtain

$$(4.9) \quad \dot{y}_m^k(t) = \lim_{\nu \rightarrow \infty} \dot{y}_{m,\nu}^k.$$

□

Recall the definition of generalized characteristics which will be used in the proof of Theorem 1.1.

Definition 4.6. A *generalized i -characteristic* associated with the approximate solution u_ν , on the time interval $[t_1, t_2] \subset [0, \infty)$, is a Lipschitz continuous function $\chi : [t_1, t_2] \rightarrow (-\infty, \infty)$ which satisfies the differential conclusion

$$\dot{\chi}(t) \in [\lambda_i(u_\nu(x+, t)), \lambda_i(u_\nu(x-, t))].$$

For any given $(T, \bar{x}) \in \mathbb{R}$, we consider the *minimal (maximal) generalized i -characteristic* through (T, \bar{x}) defined as

$$\chi^{-(+)}(t) = \min(\max)\{\chi(t) : \chi \text{ is a generalized characteristic, } \chi(T) = \bar{x}\}.$$

The properties of approximate solutions yield that there is no wave-front of i -th family crossing χ^+ from the left or crossing χ^- from the right.

Suppose $\mathcal{T}_{i,\nu} \ni y'_\nu : [t_\nu^-, t_\nu^+] \rightarrow \mathbb{R}$ and $\mathcal{T}_{i,\nu} \ni y''_\nu(\cdot) : [t_\nu^-, t_\nu^+] \rightarrow \mathbb{R}$. By Lemma 4.2 and the definition of (ϵ, i, k) -approximate subdiscontinuity curves, it turns out that either

$$y'_\nu(t) \leq y''_\nu(t), \quad \forall t \in [t_\nu^-, t_\nu^+] \cap [t_\nu^{\prime\prime-}, t_\nu^{\prime\prime+}],$$

or

$$y'_\nu(t) \geq y''_\nu(t), \quad \forall t \in [t_\nu^-, t_\nu^+] \cap [t_\nu^{\prime\prime-}, t_\nu^{\prime\prime+}].$$

This makes the following definition well defined.

Definition 4.7. Suppose $y' \in \mathcal{T}_i^{k'}(\epsilon')$, $y'' \in \mathcal{T}_i^{k''}(\epsilon'')$ where k' and k'' are both odd numbers (or even numbers), and $y'_\nu \rightarrow y'$, $y''_\nu \rightarrow y''$ as $\nu \rightarrow \infty$ and assume there exists a point (t_0, x_0) with $t_0 > 0$ such that $y'(t_0) = y''(t_0) = x_0$, we say $y' \prec y''$ if there exists a neighborhood $[t^-, t^+]$ of time t_0 such that

- $y'(t) \leq y''(t)$ for all $t \in [t^-, t^+]$,

- either there exist $t^* \in [t^-, t^+]$ such that $y'(t^*) < y''(t^*)$
or for all $t \in [t^-, t^+]$, $y'(t) = y''(t)$, and $k_1 < k_2$ ($k_1 > k_2$).

The next lemma rules out the situation when two subdiscontinuity with different sign of strength is tangent at the same point which is not the atom point of interaction and cancellation measure.

Lemma 4.8. *Suppose that $y_1 \in \mathcal{T}_i^{k_1}$, $y_2 \in \mathcal{T}_i^{k_2}$ with k_1 is even and k_2 is odd, and assume that y_1 and y_2 pass through the same point (x_0, t_0) with $\mu^{IC}(\{(x_0, t_0)\}) = 0$ and there is no subdiscontinuity curve y_0 such that*

$$y_1(t) \leq y_0(t) \leq y_2(t)$$

for any neighbourhood of t_0 . Then $\dot{y}_1(t_0) \neq \dot{y}_2(t_0)$.

Proof. Suppose $\mathcal{T}_{i,\nu}^{k_1} \ni y_{1,\nu} \rightarrow y_1$, $\mathcal{T}_{i,\nu}^{k_2} \ni y_{2,\nu} \rightarrow y_2$ and $t_{1,\nu}, t_{2,\nu} \rightarrow t_0$. Let us denote the points $y_{1,\nu}(t_{1,\nu})$, $y_{2,\nu}(t_{2,\nu})$ as A_ν, B_ν respectively. Since there is no subdiscontinuity curves between y_1 and y_2 , the strengths of all fronts crossing the segment $\overline{A_\nu B_\nu}$ tend to zero.

Moreover, the total strength of the fronts of the other family tends to zero. In fact, if not, either they are canceled in the neighborhood of (x_0, t_0) or interacted with $y_{1,\nu}(t_{1,\nu})$, $y_{2,\nu}(t_{2,\nu})$, which implies the uniform positivity of μ_ν^{IC} on a small region Γ_ν . This contradicts the assumption that $\mu^{IC}(\{(x_0, t_0)\}) = 0$. Therefore, the values of each u_ν along the segment $\overline{A_\nu B_\nu}$ remain arbitrary close to i -rarefaction curve.

On the other hand, **since the sign of the strengths of y_1 and y_2 are different**, one can always find A'_ν, B'_ν on $\overline{A_\nu B_\nu}$ and a positive constant c such that $u_\nu(A'_\nu), u_\nu(B'_\nu) \in \Delta_i^k$ for some k and $|u_\nu(A'_\nu) - u_\nu(B'_\nu)| > c$, therefore

$$|\lambda_i(u_\nu(A'_\nu)) - \lambda_i(u_\nu(B'_\nu))| \gtrsim c.$$

Up to subsequence, we can assume that for all ν

$$(4.10a) \quad \lambda_i(u_\nu(A'_\nu)) - \lambda_i(u_\nu(B'_\nu)) > c/2,$$

$$(4.10b) \quad \text{or } \lambda_i(u_\nu(A'_\nu)) - \lambda_i(u_\nu(B'_\nu)) < c/2.$$

Let us consider the case (4.10a), the other case is analogous to prove. We take χ^+ through A'_ν , χ^- through B'_ν . Since A'_ν and B'_ν are in the same Δ_i^k , if no uniformly large interaction occur on χ^+ , χ^- , they will interact with each other. We consider the region Γ_ν bounded by $\overline{A_\nu B_\nu}$, χ^+ and χ^- . Since no fronts can leave Γ_ν through χ^+ and χ^- . By (3.5) and (3.9), we obtain that $\mu^I(\Gamma_\nu) \gtrsim c$ which contradicts the assumption $\mu^{IC}(\{(x_0, t_0)\}) = 0$. \square

5. PROOF OF THEOREM 1.1

Before proving the theorem, we recall the definition space-like curve.

Definition 5.1. Let $\hat{\lambda}$ be a constant larger than the absolute value of all characteristic speed. We say a curve $x = y(t)$, $t \in [a, b]$ is *space-like* if

$$|y(t_2) - y(t_1)| > \hat{\lambda}(t_2 - t_1) \quad \text{for all } a < t_1 < t_2 < b.$$

From the definition one knows that any fronts can cross a space-like curve at most once.

Proof of Theorem 1.1. Let Θ consists of all jump points of initial data, the atom points of interaction and cancelation measure μ^{IC} and the points where two sub-discontinuity curves of different families cross each other.

Consider a point $P = (\tau, \xi) \notin \Theta$. Since $u(\cdot, \tau)$ has bounded variation, there exist the limits

$$u^L := \lim_{x \rightarrow \xi^-} u(x, \tau), \quad u^R := \lim_{x \rightarrow \xi^+} u(x, \tau).$$

Assuming that $u^R = T_i[u^L](s)$. We only consider the case for $s > 0$ and the case for $s < 0$ is analogous to prove.

Applying the tame oscillation condition (see p.295 of [3]), one obtains

$$(5.1) \quad \lim_{\substack{(x,t) \rightarrow (\xi, \tau) \\ \tau \leq t < \tau + (\xi - x)/\hat{\lambda}}} u(x, t) = u^L, \quad \lim_{\substack{(x,t) \rightarrow (\xi, \tau) \\ \tau \leq t < \tau + (x - \xi)/\hat{\lambda}}} u(x, t) = u^R.$$

for some constant $\hat{\lambda}$ which is larger than all characteristic speeds.

Suppose that there are i -subdiscontinuity curves $y_1^{k_1}, \dots, y_l^{k_l}$ satisfying $y_j^{k_j} \in \mathcal{T}_i^{k_j}(\epsilon_j)$ with $\epsilon_j > 0$, $1 \leq j \leq l$,

$$y_1^{k_1}(\tau) = \dots = y_l^{k_l}(\tau) = \xi$$

and there is no other subdiscontinuity curve passing through the point P . This can be done because of the conclusion of Lemma 4.8 and the fact that $P \notin \Theta$. It is easy to show that ϵ_j must have the same sign, i.e. $\epsilon_{j_1} \epsilon_{j_2} > 0$ for any $j_1, j_2 \in \{1, \dots, l\}$.

By rearranging the index, we can assume that

$$y_1^{k_1} \prec y_2^{k_2} \prec \dots \prec y_l^{k_l},$$

and

(H) there is no $y_0^{k_0} \in \mathcal{T}_i$ such that $y_0^{k_0} \prec y_1^{k_1}$ with $y_0^{k_0}(t_0) = y_1^{k_1}(t_0)$ or $y_l^{k_l} \prec y_0^{k_0}$ with $y_0^{k_0}(t_0) = y_l^{k_l}(t_0)$.

Step 1. By the definition, there exist $y_{1,\nu}^{k_1}, \dots, y_{l,\nu}^{k_l} \in \mathcal{T}_{i,\nu}$ such that $y_{j,\nu}^{k_j} \rightarrow y_j^{k_j}$, $\forall j \in \{1, \dots, l\}$. We claim that

$$(5.2a) \quad \lim_{r \rightarrow 0+} \limsup_{\nu \rightarrow \infty} \left(\sup_{\substack{x < y_{1,\nu}^{k_1}(t) \\ (x,t) \in B(P,r)}} |u_\nu(x,t) - u^L| \right) = 0,$$

$$(5.2b) \quad \lim_{r \rightarrow 0+} \limsup_{\nu \rightarrow \infty} \left(\sup_{\substack{x > y_{l,\nu}^{k_l}(t) \\ (x,t) \in B(P,r)}} |u_\nu(x,t) - u^R| \right) = 0,$$

where $B(P,r)$ is a ball centred at P with radius r .

Indeed, if (5.2a) is not true, by the first limit in (5.1) and $u_\nu \rightarrow u$ pointwise a.e., there exist two sequences of points P_ν, Q_ν converging to P and P_ν, Q_ν on the left of $y_1^{k_1}$ such that the segment $\overline{P_\nu Q_\nu}$ is space-like and

$$u(P_\nu) \rightarrow u^L$$

and

$$|u_\nu(P_\nu) - u_\nu(Q_\nu)| \geq \epsilon_0.$$

It is not restrictive to assume that the direction $\overrightarrow{P_\nu Q_\nu}$ towards $y_{1,\nu}^{k_1}$.

Let $\Lambda_j(\overline{P_\nu Q_\nu})$ be the total wave strength of wave-fronts of j -th family which cross the segment $\overline{P_\nu Q_\nu}$. Then, one has $\Lambda_j(\overline{P_\nu Q_\nu}) \gtrsim \epsilon_0$ for some $j \in \{1, \dots, d\}$. We consider the following three cases.

Case 1. If $j > i$, we take the maximal forward generalized j -characteristic χ^+ through P_ν and minimal generalized j -characteristic χ^- through Q_ν .

If χ^+ and χ^- interact each other at O_ν before hitting $y_{1,\nu}^{k_1}$. We consider the region Γ_ν bounded by $\overline{P_\nu Q_\nu}$, χ^+ and χ^- . Since no fronts can leave Γ_ν through χ^+ and χ^- . By (3.5) and (3.9), we obtain that the amount of interaction and cancellation $\mu_\nu^{IC}(\bar{\Gamma}_\nu)$ for u_ν within the closure of Γ_ν remains uniformly positive as $\nu \rightarrow \infty$.

If χ^+ interacts $y_{1,\nu}^{k_1}$ at A_ν and χ^- interacts $y_{1,\nu}^{k_1}$ at B_ν , we consider the region Γ_ν bounded by $\overline{P_\nu Q_\nu}$, χ^+ , χ^- and $y_{1,\nu}^{k_1}$. Then either there exists a constant $0 < c'_0 < 1$ such that $\mu_\nu^{IC}(\Gamma_\nu) > c'_0 \epsilon_0$ or there exists a constant $0 < c''_0 < 1$ such that fronts with total strength larger than $c''_0 \epsilon_0$ hitting $\overline{A_\nu B_\nu}$. By (3.5) and (3.9) we can determine that $\mu_\nu^{IC}(\bar{\Gamma}_\nu) \gtrsim \epsilon$ uniformly.

For both above two cases, Γ_ν is contained in a ball $B(P, \tilde{r}_\nu)$ with $\tilde{r}_\nu \rightarrow 0$ as $\nu \rightarrow \infty$, which implies that $\mu^{IC}(\{P\}) > 0$. This is against the assumption $P \notin \Theta$.

Case 2. If $j < i$, we consider the minimal backward generalized j -characteristic through the point P_ν and the maximal backward generalized j -characteristic through the point Q_ν . Then by the similar argument for the case $j > i$, we get $\mu^{IC}(\{P\}) > 0$ against the assumptions.

Case 3. If $j = i$ and for any $j' \neq i$, $1 \leq j' \leq d$, $\Lambda_{j'}(\overline{P_\nu Q_\nu}) \rightarrow 0$ as $\nu \rightarrow \infty$. We claim that the maximum of the strengths of all fronts which cross $\overline{P_\nu Q_\nu}$ tends to zero when $\nu \rightarrow \infty$.

If it is not true, since $\Lambda_{j'}(\overline{P_\nu Q_\nu}) \rightarrow 0$ for $j' \neq i$, there must be fronts of i -th family across $\overline{P_\nu Q_\nu}$ with uniformly large strength. We assume that up to a subsequence, their (i, k_0) -substrength $s_\nu^{k_0}$ are uniformly large for some $k_0 \in \{1, \dots, \bar{k}_i\}$, that is $|s_\nu^{k_0}| \gtrsim \epsilon$ for some $\epsilon > 0$.

Then by Definition 4.3 there must be, for some $\epsilon_0 > 0$, (ϵ_0, i, k_0) -approximate subdiscontinuity curves $y_{0,\nu}^{k_0}$ which contains the wave fronts $s_\nu^{k_0}$ (we use the same notation as their strength for convenience) and since $y_{0,\nu}^{k_0}$ are uniformly Lipschitz continuous curves, up to a subsequence, there is a Lipschitz continuous curves $y_0^{k_0}$, such that

$$y_{0,\nu}^{k_0} \rightarrow y_0^{k_0} \in \mathcal{T}_i^{k_0}, \quad \nu \rightarrow \infty$$

and $y_0^{k_0}(\tau) = \xi$. By Definition 4.7 we obtain $y_0^{k_0} \prec y_1^{k_1}$, which contradicts the assumption (H).

So we can always choose $Q'_\nu, P'_\nu \in \overline{P_\nu Q_\nu}$ such that

$$u_\nu(Q'_\nu) \rightarrow u^L$$

and

$$|u_\nu(Q'_\nu) - u_\nu(P'_\nu)| \geq c_0 \epsilon_0,$$

where $0 < c_0 < 1$ and $u_\nu(Q'_\nu), u_\nu(P'_\nu)$ locate in the same Δ_i^k for some k .

Since for $j' \neq i$, $\Lambda_{j'}(\overline{P_\nu Q_\nu})$ is arbitrary small when ν is large enough and the strength of fronts belonging to i -th family is small, one has

$$\max_{H_\nu \in \overline{P'_\nu Q'_\nu}} \min_{s^* \geq 0} |u_\nu(H_\nu) - R_i[u_\nu(P'_\nu)](s^*)| \ll 1.$$

Which means the values of each u_ν along the segment $\overline{P'_\nu Q'_\nu}$ remain arbitrary close to the i -rarefaction curve through u^L . Then by the analogous argument in the proof of Lemma 4.8, one gets the contradiction $\mu^{IC}(\{P\}) > 0$. Therefore, we conclude that (5.2) is true. And (5.2b) is similar to prove.

Step 2. Define $\mathcal{T} = \bigcup_i \mathcal{T}_i$. If $P \notin \Theta \cap \text{Graph}(\mathcal{T})$ and if u is not continuous at P , then there exist $\epsilon > 0$ and $P_\nu, Q_\nu \rightarrow P$ such that $\overline{P_\nu Q_\nu}$ is space like and

$$u_\nu(P_\nu) \rightarrow u(P), \quad |u_\nu(Q_\nu) - u(P)| \geq \epsilon \text{ for all } \nu.$$

Up to subsequence, we consider the following two cases.

- 1) there exists $j \neq j'$, such that $\min\{\Lambda_j(\overline{P_\nu Q_\nu}), \Lambda_{j'}(\overline{P_\nu Q_\nu})\} \geq \epsilon_0$. This situation can be ruled out by the argument in Case 1 of Step 1.
- 2) For some $j \in \{1, \dots, j\}$ and all ν , $\Lambda_j(\overline{P_\nu Q_\nu}) \geq \epsilon_0$ and for all $j' \neq j$, $\Lambda_{j'}(\overline{P_\nu Q_\nu}) \rightarrow 0$ as $\nu \rightarrow \infty$. Then one can use the argument in Case 3 of Step 1 to obtain the contradiction.

Therefore we get the continuity of u outside $\text{Graph}(\mathcal{T}) \cup \Theta$.

Then, we have proved the first part of the theorem.

Step 3. We now establish the Rankine-Hugoniot condition (1.5) for curves in \mathcal{T} . Let $P = (t_0, x_0) \in \text{Graph}(\mathcal{T}) \setminus \Theta$, and write

$$u^L = \lim_{x \rightarrow x_0^-} u(x, t_0), \quad u^R = \lim_{x \rightarrow x_0^+} u(x, t_0).$$

We consider two cases.

Case 1. There is only one curve $y \in \mathcal{T}_i$ passing through point P . From (5.2), we know that the discontinuity $[u^L, u^R]$ must be simple. Suppose that $\mathcal{T}_{i,\nu} \ni y_\nu \rightarrow y$ as $\nu \rightarrow \infty$. By (3.3) and (3.4), we obtain

$$\begin{aligned} \sigma_i(u_\nu^L, u_\nu^R)[u_\nu^L - u_\nu^R] &= f(u_\nu^L) - f(u_\nu^R), \\ |y_\nu - \sigma_i(u_\nu^L, u_\nu^R)| &< 2\epsilon_\nu, \end{aligned}$$

where

$$u_\nu^L = \lim_{x \rightarrow y_\nu(t_\nu)^-} u(t_\nu, x), \quad u_\nu^R = \lim_{x \rightarrow y_\nu(t_\nu)^+} u(t_\nu, x), \quad t_\nu \rightarrow t_0 \text{ as } \nu \rightarrow \infty.$$

Then by (5.2), one has for every $\epsilon > 0$ there exists $\bar{\nu}(\epsilon)$ such that $\forall \nu > \bar{\nu}$, one has

$$|y_\nu - \sigma_i(u^L, u^R)| \leq |y_\nu - \sigma_i(u_\nu^L, u_\nu^R)| + |\sigma_i(u^L, u^R) - \sigma_i(u_\nu^L, u_\nu^R)| < \epsilon.$$

From (4.9) and the fact that the i -waves of Riemann problem constructed by Theorem 2.1 are Liu admissible, one deduces that

$$\dot{y}(t_0)(u^R - u^L) = f(u^R) - f(u^L),$$

and

$$\dot{y}(t_0) \leq \hat{\sigma}_i(S_i[u^L](\tau), u^L), \quad \forall \tau \in [0, s].$$

Case 2. If the discontinuity $[u^L, u^R]$ is a composition of $[u_0, u_1], [u_1, u_2], \dots, [u_l, u_{l+1}]$ where $u_0 = u^L, u_{l+1} = u^R$, $u_j = T_i[u^L](s_j)$ and $u^L \in \Delta_i^{k_1^*}, u^R \in \Delta_i^{k_2^*}$. Let

$$\begin{aligned} k_1 &= \min\{k \text{ is even}, k \geq k_1^*\}, \\ k_p &= \max\{k \text{ is even}, k \leq k_2^*\} \end{aligned}$$

and $k_p = k_1 + 2(p-1)$. One has $p \geq l+1$.

According to (5.2), there exist $P_\nu, Q_\nu \rightarrow P$ such that

$$u_\nu(P_\nu) \rightarrow u^L, \quad u_\nu(Q_\nu) \rightarrow u^R,$$

and the segment $\overline{P_\nu Q_\nu}$ is space-like.

We claim that there exist p subdiscontinuity curves $y_1, \dots, y_p \in \mathcal{T}_i$ passing through point P , where $y_j \in \mathcal{T}_i^{k_1+2j}$, $j = 1, \dots, p-1$.

In fact, let $\mathcal{S}_i^k(\overline{P_\nu Q_\nu})$ denote the maximal (i, k) -substrength of all fronts across $\overline{P_\nu Q_\nu}$. It is sufficient to show that, up to a subsequence, there is a constant $C > 0$ such that

$$\mathcal{S}_i^k(\overline{P_\nu Q_\nu}) \geq C, \quad \forall k \in \{k_1, \dots, k_p\}.$$

If not, there exists $k_0 \in \{k_1, \dots, k_p\}$ such that $\mathcal{S}_i^{k_0}(\overline{P_\nu Q_\nu}) \rightarrow 0$ as $\nu \rightarrow \infty$. Since $\mu^{IC}(P) = 0$, we have

$$\Lambda_j(\overline{P_\nu Q_\nu}) \rightarrow 0, \quad \text{as } \nu \rightarrow \infty, \quad \forall j \neq i.$$

By Lemma 4.8, we conclude that $\mathcal{S}_i^k(\overline{P_\nu Q_\nu}) \rightarrow 0$, as $\nu \rightarrow \infty$, $\forall k$ odd. Since the (i, k_0) -substrength of all wave fronts are arbitrary small, up to a subsequence, one can always find points P'_ν, Q'_ν on $\overline{P_\nu Q_\nu}$ such that $u_\nu(P'_\nu), u_\nu(Q'_\nu) \in \Delta_i^{k_0}$ and all fronts are either admissible discontinuity with left and right value inside $\Delta_i^{k_0}$ or rarefaction fronts. Therefore by the analogous argument in Case 3 of Step 1, there exist a constant $c > 0$ independent on ν such that in a small neighborhood Γ_ν of P , one has $\mu_\nu^{IC}(\Gamma_\nu) \geq c$. This contradicts with the assumption $\mu^{IC}(P) = 0$. This concludes our claim.

Moreover, by Lemma 4.8 and the equalities (5.2), there are exactly p subdiscontinuity curves passing through P , otherwise P must be an interaction point with $\mu^{IC}(P) > 0$.

Suppose $u_{n+1} = T_i[u_n](s_n)$ for some $s_n > 0$, $n \in \{0, \dots, l\}$ and $T_i[u_n](\cdot)$ intersects $Z_i^{k_{n_1}}, \dots, Z_i^{k_{n_q}}$ at $u_{n,1}, \dots, u_{n,q}$. Then the subdiscontinuity curves with substrength $s_i^{k_{n_1}}, \dots, s_i^{k_{n_q}}$ must coincide in neighborhood of time τ .

If in a neighborhood of P , y_j and y_{j+1} are not identical, by the similar argument for proving (5.2), one can show that (1.8) is true.

Suppose $\mathcal{T}_{i,\nu} \ni y_{j,\nu} \rightarrow y_j$, $j \in \{1, \dots, p\}$. As we discussed above, it is not restrictive to assume that $p = l+1$ and there is a neighborhood $U(t_0)$ of t_0 , such that $\forall t \in U(t_0)$,

$$y_{1,\nu}(t) < \dots < y_{p,\nu}(t).$$

Then, using the similar argument in Step 1, besides (5.2) and (5.2b), one can also show that

$$\begin{aligned} \lim_{r \rightarrow 0+} \limsup_{\nu \rightarrow \infty} \left(\sup_{\substack{x < y_{m,\nu}^{k_m}(t) \\ (x,t) \in B(P,r)}} |u_\nu(x,t) - u_{m-1}| \right) &= 0 \quad m = 2, \dots, p, \\ \lim_{r \rightarrow 0+} \limsup_{\nu \rightarrow \infty} \left(\sup_{\substack{x > y_{n,\nu}^{k_n}(t) \\ (x,t) \in B(P,r)}} |u_\nu(x,t) - u_n| \right) &= 0, \quad n = 1, \dots, l. \end{aligned}$$

For notational convenience, we write $u_0 = u^L$, $u_p = u^R$. Therefore, by the same argument in Case 1, we obtain that for $n = 1, \dots, p$

$$\dot{y}_n(t_0)(u_n - u_{n-1}) = f(u_n) - f(u_{n-1})$$

and

$$\dot{y}_n(t_0) \leq \hat{\sigma}_i(u_n, S_i[u^L](\tau)), \quad \forall \tau \in [s_{j-1}, s_j].$$

Adding them together, one finally obtain for $m = 1, \dots, p$

$$\dot{y}_m(t_0)(u^R - u^L) = f(u^R) - f(u^L)$$

and

$$\dot{y}_m(t_0) \leq \hat{\sigma}_i(u^L, S_i[u^L](\tau)), \quad \forall \tau \in [0, s].$$

□

The proof of Theorem 1.1 already implies the results of Theorem 1.2.

Concerning a sequence of exact solution of (1.1) such that $u_\nu \rightarrow u$ in L^1_{loc} , one can approximate each u_ν by a sequence of wave-front tracking approximations $u_{m,\nu} \rightarrow u_\nu$ and by a suitable diagonal sequence $u_{\nu,m(\nu)} \rightarrow u$ one has the following corollary.

Corollary 5.2 (stability of discontinuity curves). *Considering a sequence of exact solutions u_ν such that $u_\nu \rightarrow u$ in L^1_{loc} , one has*

- (1) *Let $y_\nu : [t_\nu^-, t_\nu^+] \mapsto \mathbb{R}$ be a discontinuity curves of u_ν described in Theorem 1.1. Assume $t_\nu^- \rightarrow t^-, t_\nu^+ \rightarrow t^+$ and $y_\nu(t) \rightarrow y(t)$ for each $t \in [t^-, t^+]$. Then $y(\cdot)$ is a discontinuity curve of the limiting solution u with the properties mentioned in Theorem 1.1.*
- (2) *Viceversa, let $y : [t^-, t^+] \mapsto \mathbb{R}$ be a discontinuity curve of u for a.e. $t \in [t^-, t^+]$. Then there exists a sequence of discontinuity curves $y_\nu : [t_\nu^-, t_\nu^+] \mapsto \mathbb{R}$ of u_ν such that*

$$t_\nu^- \rightarrow t^-, \quad t_\nu^+ \rightarrow t^+, \quad \lim_{\nu \rightarrow \infty} y_\nu(t) = y(t),$$

for a.e. $t \in [t^-, t^+]$.

6. AN REMARK ON GENERAL STRICT HYPERBOLIC SYSTEMS

We construct a strict hyperbolic system of conservation laws with one characteristic family which is not linearly degenerate or piecewise genuinely nonlinear. Therefore neither the assumption of Theorem 1.1 or that of Theorem 10.4 in [6] holds. We show that the set of jump points of its admissible solution to some initial data can not be “exactly” covered by countably many Lipschitz continuous curves.

Consider the following 2×2 system

$$(6.1) \quad \begin{cases} u_t + f(u, v)_x = 0, \\ v_t - v_x = 0. \end{cases}$$

where f is a smooth function and u, v is the unknown variables. The Jacobian matrix of flux function is

$$DF(u, v) = \begin{pmatrix} f_u & f_v \\ 0 & -1 \end{pmatrix}.$$

Then the eigenvalues are

$$\lambda_1 = -1, \quad \lambda_2 = f_u.$$

And the corresponding right eigenvectors are

$$r_1(u, v) = (f_v, -f_u - 1)^T, \quad r_2 = (1, 0)^T.$$

The system is strict hyperbolic if $f_u > -1$. (In fact, f constructed latter satisfies this property). Obviously, one has

$$Z_1 = \{(u, v) : \nabla \lambda_1 \cdot r_1(u, v) = 0\} = \mathbb{R}^2,$$

which means that the 1-th characteristic family is linearly degenerate.

Latter we will also show that

$$(6.2) \quad Z_2 = \{(u, v) : \nabla \lambda_2 \cdot r_2\} = \{(u, v) : f_{uu}(u, v) = 0\} = \{v = 0\}.$$

This yields that the vector field r_2 is tangent to the manifold Z_2 , therefore the second characteristic family is not piecewise genuinely nonlinear or linearly degenerate..

Define $f(u, v) = e^{-1/v} u^2 / 2$ when $v > 0$ and $f(u, 0) \equiv 0$. In the following, we define the value of f for $v < 0$.

Let the initial data to be

$$(6.3) \quad u_0(x) = \begin{cases} u_l & x < 0, \\ u_r & x > 0, \end{cases} \quad v_0(x) = \begin{cases} -a & x < h, \\ a & x > h, \end{cases}$$

for some small constants $u_l > u_r$ and $a, h > 0$.

Since the second equation in (6.1) is a linear transport equation, one has

$$(6.4) \quad v(x, t) = \begin{cases} -a & x + t < h, \\ a & x + t > h. \end{cases}$$

Then one can solve the system (6.1) by regarding it as a scalar conservation laws of u

$$u_t + f(u, v)_x = 0$$

with discontinuous coefficient when v is a piecewise constant function.

If w.r.t. u , f is concave for some small fixed $v < 0$ and concave for some fixed $v > 0$, then u is a center rarefaction wave in the area $\{x + t < h\}$. Immediately after the rarefaction wave crosses the characteristic line $x + t = h$, it turns out to be a compressive wave since $f(u, -a)$ is a convex function on u . It may generate a shock after a short time.

Indeed, this can be done in the following way.

Consider a center rarefaction wave of u in the area $\{x + t < h\}$

$$(6.5) \quad u(x, t) = \begin{cases} u_l & x/t < f_u(u_l, -a), \\ g(x/t) & x/t = f_u(g(x/t), -a), \\ u_r & x/t > f_u(u_r, -a), \end{cases}$$

where $g(\cdot)$ is the converse function of $f_u(\cdot, -a)$.

Let $u^- = g(\xi^*)$ for some ξ^* . The value of u will be u^- along the characteristic line

$$x = t f_u(u^-, -a)$$

until it intersects another characteristic line $x + t = h$.

Solving the equations

$$(6.6) \quad \begin{cases} x = t f_u(u^-, -a), \\ x = -t + h, \end{cases}$$

we get the intersection point of two characteristics:

$$(6.7) \quad \begin{cases} x_0 = h f_u(u^-, -a) / [1 + f_u(u^-, -a)], \\ t_0 = h / [1 + f_u(u^-, -a)]. \end{cases}$$

Next, we compute the value of u after the characteristic cross the line $x + t = h$. For each point (x_0, t_0) on the line $x + t = h$, let

$$u^+(x_0, t_0) = \lim_{\substack{x+t > h \\ (x,t) \rightarrow (x_0, t_0)}} u(x, t).$$

By Rankine-Hugoniot condition, one has

$$-(u^+ - u^-) = \frac{e^{-1/a}(u^+)^2}{2} - f(u^-, -a).$$

This yields

$$(6.8) \quad u^+ = \frac{-1 + \sqrt{1 + 2e^{-1/a}(f(u^-, -a) + u^-)}}{e^{-1/a}}.$$

Since $f(u, v) = e^{-1/a}u^2/2$ on area $\{h < x + t < 2h\}$, the characteristic line of the equation

$$u_t + f(u, a)_x = 0$$

starting at (x_0, t_0) is

$$x - \frac{h f_u(u^-, -a)}{1 + f_u(u^-, -a)} = e^{-1/a} u^+ \left(t - \frac{h}{1 + f_u(u^-, -a)} \right).$$

We require that it passes through the point $(0, 2h)$, that is

$$-\frac{h f_u(u^-, -a)}{1 + f_u(u^-, -a)} = e^{-1/a} u^+ \left(2h - \frac{h}{1 + f_u(u^-, -a)} \right),$$

We can replace u_0 in the initial data by

$$\tilde{u}_0 = \begin{cases} u_l & x < 0, \\ u_r & 0 < x < h, \\ u_1 & x > h \end{cases}$$

to make the speed of the characteristics starting from $\{h < x < 2h\}$ the same as nearby. By the total variation estimates for the general system

$$\text{Tot.Var.}\{u(\cdot, t)\} \lesssim \text{Tot.Var.}\{u_0(\cdot)\},$$

it is not restrict to assume that the total variation of \tilde{u}_0 is sufficiently small.

Now, we begin to find the initial data with which (6.1) had an admissible solution containing “Cantor set” shocks.

Now, we consider the initial data

$$(6.11) \quad u_0(x) = \begin{cases} u_l & x < 0, \\ u_r & x > 0, \end{cases} \quad v_0(x) = \begin{cases} 0 & x < 2h, \\ -a & 2h < x < 3h, \\ a & 3h < x < 4h, \\ 0 & x > 4h. \end{cases}$$

Since the second equation in (6.1) is a linear transport equation, one has

$$(6.12) \quad v(x, t) = \begin{cases} 0 & x + t < 2h, \\ -a & 2h < x + t < 3h, \\ a & 3h < x + t < 4h, \\ 0 & x + t > 4h. \end{cases}$$

In the area $\{x + t < 2h\}$, the first equation in (6.1) becomes $u_t = 0$, then one has

$$(6.13) \quad u(x, t) = \begin{cases} u_l & x < \min\{2h - t, 0\}, \\ u_r & x > \max\{2h - t, 0\} \end{cases}$$

Next, we compute the value of u in $\{2h < x + t < 3h\}$. For any $(x', t') \neq (0, 2h)$ on the line $x + t = 2h$, write

$$u^- = \lim_{\substack{(x,t) \rightarrow (x',t') \\ x+t < 2h}} u(x, t), \quad u^+ = \lim_{\substack{(x,t) \rightarrow (x',t') \\ x+t > 2h}} u(x, t).$$

By Rankine-Hugoniot conditions, one has

$$f(u^+, -a) - f(u^-, 0) = -(u^+ - u^-).$$

which yields

$$u^- = u^+ + f(u^+, -a).$$

Regarding u^+ as a function of u^- and differentiating the above equation on both sides with respect to u^- , one gets

$$(f_u + 1)(u^+)' = 1.$$

By (6.9), one has

$$(u^+)' = \frac{1}{1 + f_u} = \frac{2g - 1}{g} > 0$$

in a small neighborhood of origin. Thus u^+ is strictly increasing w.r.t. u^- . This concludes that at point $(0, 2h)$ the left value of u is still larger than the right value of u , i.e.

$$u_l^+ > u_r^+,$$

where

$$u_l^+ = \lim_{\substack{(x,t) \rightarrow (0,2h) \\ 2h-t < x < 0}} u(x, t), \quad u_r^+ = \lim_{\substack{(x,t) \rightarrow (0,2h) \\ x > \max\{2h-t, 0\}}} u(x, t).$$

As we discussed before, one gets a rarefaction wave by solving the Riemann problem in $\{2h < x + t < 3h\}$ and a compressive waves in area $\{3h < x + t < 4h\}$.

Next, we compute the value of u in the zone $\{x + t > 4h\}$. For any point $(x_0, t_0) \neq (0, 4h)$ on the line $x + t = 4h$, let

$$u^+(x_0, t_0) = \lim_{\substack{x+t > 4h \\ (x,t) \rightarrow (x_0, t_0)}} u(x, t), \quad u^-(x_0, t_0) = \lim_{\substack{x+t < 4h \\ (x,t) \rightarrow (x_0, t_0)}} u(x, t).$$

By Rankine-Hugoniot conditions, one has

$$f(u^+, 0) - f(u^-, a) = -(u^+ - u^-).$$

Then since $f(u^+, 0) = 0$ and $f(u^-, a) = e^{-1/a}(u^-)^2/2$, one obtains

$$u^+ = \frac{e^{-1/a}(u^-)^2}{2} + u^-.$$

We claim that $u_L^+ > u_R^+$, where

$$u_L^+ = \lim_{\substack{(x,t) \rightarrow (4h,0) \\ 4h-t < x < 0}} u(x, t), \quad u_R^+ = \lim_{\substack{(x,t) \rightarrow (4h,0) \\ x > \max\{4h-t, 0\}}} u(x, t).$$

are the left/right of u across the line $x + t = 4h$. In fact, since $u_L^- > u_R^-$, where

$$u_L^- = \lim_{\substack{(x,t) \rightarrow (4h,0) \\ x+t < 4h \\ x < \hat{\lambda}(t-4h)}} u(x, t), \quad u_R^- = \lim_{\substack{(x,t) \rightarrow (4h,0) \\ x+t < 4h \\ x > -\hat{\lambda}(t-4h)}} u(x, t).$$

one has

$$\begin{aligned} & u_L^+ - u_R^+ \\ &= \left(\frac{e^{-1/a}(u_L^-)^2}{2} + u_L^- \right) - \left(\frac{e^{-1/a}(u_R^-)^2}{2} + u_R^- \right) \\ &= (u_L^- - u_R^-) \left(1 - \frac{e^{-1/a}(u_L^-)^2}{2} (u_L^- + u_R^-) \right) > 0 \end{aligned}$$

since a and (u_L^-, u_R^-) sufficiently small.

Therefore as the first equation in (6.1) is $u_t = 0$ in $\{x + t > 4h\}$, the jump propagate along $x = 0$ which turns out to be a shock. Similarly, by modifying the initial data a little, we can guarantee that there is not other shocks in the solution. (See Figure 4.)

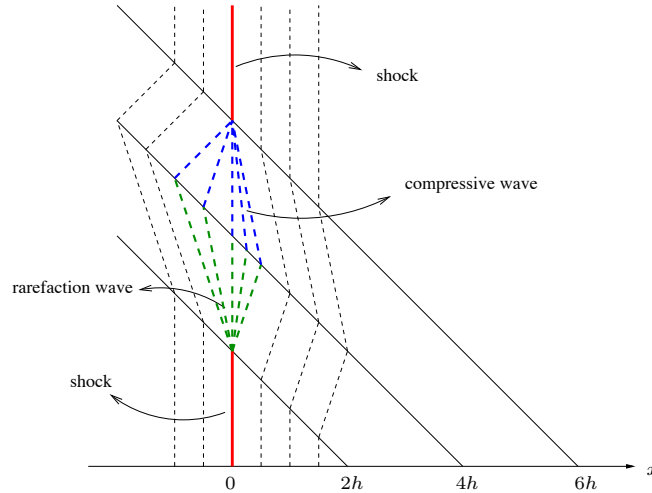


FIGURE 4. After modifying a little the initial data (6.11), the admissible solution contains two shocks along the t -axis.

Next, we construct an admissible solution such that it consists of graph of shock curves which is an 1-dimensional Cantor set. Let

$$B_{m,n}^- = \left(\frac{3n+1}{3^m}, \frac{3n+3/2}{3^m} \right) \cdot 6, \quad B_{m,n}^+ = \left(\frac{3n+3/2}{3^m}, \frac{3n+2}{3^m} \right) \cdot 6, \quad n = 0, 1, \dots, 3^{m-1} - 1.$$

and

$$B_{m,n} = B_{m,n}^- \cup B_{m,n}^+ \quad \text{and} \quad B_m = \bigcup_{n=0}^{3^{m-1}-1} B_{m,n}.$$

We consider initial datum as

$$u_{0,m}(x) = \begin{cases} u_l & x < 0, \\ u_r & x > 0, \end{cases} \quad v_{0,m} = \begin{cases} a_n & x \in B_{m,n}^-, \\ -a_n & x \in B_{m,n}^+, \\ 0 & x \in \mathbb{R} \setminus B_{m,n}, \end{cases}$$

where one can always choose u_l , u_r and a suitable sequence $\{a_n\}$ such that total variation of (u_0, v_0) is sufficiently small. By modifying u_0 properly similarly to get rid of extra shocks. One finds that the admissible solution (u_m, v_m) of the system (6.1) is continuous inside \hat{K}_m where

$$\hat{K}_m := \bigcup_{m=1}^{\infty} \bigcup_{n=0}^{3^{m-1}-1} \{(x, t) \in \mathbb{R}^+ \times \mathbb{R} : \frac{6(3n+1)}{3^m} - x < t < \frac{6(3n+2)}{3^m} - x\}$$

and its shocks belongs to 1-th family are located on

$$\{(0, t) : t \in [0, \infty)\} \setminus \bigcup_{k=0}^{3^{m-1}-1} \left\{ (0, t) : \frac{6(3k+1)}{3^m} < t < \frac{6(3k+2)}{3^m} \right\}.$$

Obviously, $(u, v) := \lim_{m \rightarrow \infty} (u_m, v_m)$ is the admissible solution of (6.1) with initial data $(u_0, v_0) := \lim(u_{0,m}, v_{0,m})$. As $m \rightarrow \infty$, $C_m := [0, 6] \setminus B_m$ converge to the Cantor set C (scaled by 6). Since the Cantor set is uncountable set and can not contain any interval of non-zero length, it is impossible to find countable many Lipschitz continuous curves which exactly cover discontinuities of (u, v) , that is, on these curves, either (u, v) is discontinuous or it is an interaction point. This means that Theorem 1.1 fails for this situation.

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